EXTEMDING THE CLASS OF SOLUTIONS OF THE DIRAC EQUATION USING THE TRIDIAGONAL MATRIX REPRESENTATIONS

BY

IBSAL A. T. ASSI

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This thesis, written by **Ibsal A. T. Assi** under the direction his thesis advisor and approved by his thesis committee, has been presented and accepted by the Dean of Graduate Studies, in partial fulfillment of the requirements for the degree of **MASTER OF SCIENCE IN PHYSICS.**

Dr. Hocine Bahlouli
(Advisor)

Dr. Abdullah Al-Sunaidi

Department Chairman

Dr. Abdulaziz Alhaidari
(Member)

Dr. Salam A. Zummo
Dean of Graduate Studies

Dr. Saeed Al-Marzoug
(Member)

12/4/16

Date

Dedication

To my beloved family for their unfailing support and continuous encouragement

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LIST OF ABBREVIATIONS

SIB	:	Square Integrable Basis
TMRA	:	Tridiagonal Matrix Representations Approach
HEP	:	High Energy Physics
KBR	:	Kinetic Balance Relation
ODE	:	Ordinary Differential Equation

ABSTRACT

Full Name : [Ibsal A. T. Assi]

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This research is mainly focused on extending the class of solvable potentials for the 1D Dirac equation using the tridiagonal matrix representation approach. We write the spinor wavefunction as an infinite sum in a square integrable spinor basis set in which the expansion coefficients are functions of energy and potential parameters. The kinetic balance relation is used to eliminate the lower basis component which is really believed to be a good way to avoid the spurious modes of Dirac equation. Restricting the wave operator to be tridiagonal and symmetric transforms the wave equation into a three-term recursion relation of the expansion coefficients. Moreover, this recursion relation can be solved either by comparison to a well-known class orthogonal polynomials, whenever possible, or could be a new class of polynomials, in both cases the recursion relation is exactly solvable. Solving the recursion relation will provide us with full details about the eigenstates and the energy spectrum of the relativistic system. As an illustration, we considered several solvable potentials in both Laguerre and Jacobi bases. In some cases we succeeded in solving the recursion relation and wrote the solutions in closed form while in other cases we could not, but did show explicitly few of these polynomials.

ملخص الرسالة

الاسم الكامل: إبسال عادل ذياب عاصي

عنوان الرسالة: توسعة مجموعة الحلول لمعادلة ديراك بإستخدام التمثيل ثلاثي القطر لمصفوفة الموجة

التخصص: فيزياء

تاريخ الدرجة العلمية: ماجستير

نسعى في هذا البحث الى توسعة مجموعة الجهود التي يمكن حلها لمعادلة ديراك ذات البعد الواحد والتي يمكن حلها باستخدام طريقة تمثيل المصفوفة ثلاثية القطر لمعادلة الموجة. الفكرة الأساسية لهذه الطريقة هو أننا نقوم بكتابة دالة موجة المغزل على شكل متسلسلة لانهائية بحيث تكون معاملات الحدود تعتمد على الطاقة وثوابت الجهد. وأما الحدود فتكون عبارة عن إقترانات مربعة التكامل تعمد على موقع الجسم فقط. لقد تم إستخدام علاقة التوازن الحركي بين مركبات دالة الموجة واللتي يعتقد أنها طريقة مناسبة لتجنب الحلول الزائفة لمعادلة ديراك. إذا تم تقييد مصفوفة الموجة لتكون ثلاثية القطر ومتماثلة فإن ذلك ينتج عنه تحويل معادلة الموجة إلى علاقة تكرارية ثلاثية الحدود لمعاملات المتسلسلة. حل هذه العلاقة التكرارية يمكن أن يكون من خلال مقارنة بأحد العلاقات التكرارية لاقترانات عمودية معروفة أو حل جديد بأستخدام طرق متعددة. حل هذه المعادلة يعطي معلومات كاملة عن دالة الموجة وطيف الطاقة. تمكنا في هذا البحث من توسعة مجموعة الحلول لمعادلة ديراك وقمنا بذكر أمثلة متعددة بعضها تم حل المعادلة التكرارية لها والبعض الآخر كانت العلاقات التكرارية جديدة ، كما قمنا بعرض بعض الحلول لهذه العلاقات.

CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

Since the early days of quantum physics, exactly solvable potentials were of great interest for a better understanding of the quantum behavior of physical systems. Moreover, these solutions can be used to test the accuracy of perturbative and numerical approximations that were used to study certain quantum system. In fact, in some cases these solutions may constitute an analytical solution to a realistic physical problem or an approximation thereof. Finding the exact solutions to the wave equation gives full details about its eigenfunctions and the corresponding energy spectrum [1, 2]. In this work, we will use the tridiagonal representations approach, sometimes called the J-matrix inspired approach, to extend the class of solvable potentials of Dirac equation in 1+1 (space-time) dimensions [3].

This chapter is divided into two main sections. The first section talks about the development of Dirac equation and shows its importance in physics. It also gives the most general form of Dirac equation in 1+1 dimension which will be the main problem we are going to solve in this research. On the other hand, the next section gives details about the tridiagonal matrix representation approach (TMRA) which will be used here to solve Dirac equation. More details and the full mathematical derivations are given in the appendices to the interested readers.

1.1 Development of Dirac Equation

In nonrelativistic quantum mechanics, the dynamics of quantum particles is described by Schrodinger equation which reads in one spacial dimension as:

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t)=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t)+V(x)\Psi(x,t), \tag{1.1}$$

Its origin is based on energy conservation, that is, $H=\frac{p^2}{2m}+V=E$ and the operator forms of both momentum and energy which are $p\rightarrow -i\hbar\hat{\nabla}, E\rightarrow i\hbar\frac{\partial}{\partial t}$, respectively.

To write the relativistic form of Eq. (1.1), we should write down the conservation of energy described by Einstein's equation in relativistic units where $\hbar=c=1$ [1]:

$$E^2=m^2+p^2, \tag{1.2}$$

Using the operator form of the momentum and the total energy given above, we can write down the relativistic form of (1.1) which is called the Klein-Gordon equation as follows [2]:

$$\frac{\partial^2}{\partial x^2}\Psi(x,t)-\frac{\partial^2}{\partial t^2}\Psi(x,t)=m^2\Psi(x,t), \tag{1.3}$$

This equation is suitable for spinless particles only. If one tries to use the plane wave solution of (1.3) to find the probability density, the result will be that this density is directly proportional to the energy which gives a negative probability density for negative energy values. This, together with other problems especially the second derivative in time made Dirac think about ways to factorize this equation and make it first order in time [2].

Dirac equation is a relativistic wave equation that describes the dynamics of spin ½ particles such as electrons and protons, which was derived by Paul Dirac in 1928. Moreover, this equation predicts the existence of antimatter theoretically which was observed experimentally few years later after this discovery. The origin of this equation is based on both symmetry and energy considerations (positive and negative energies) [4]. After some algebraic manipulations, Dirac equation in 3+1 (space-time) dimensions for free spinor particle takes the following form:

$$\left\{i\gamma^{\mu}\partial_{\mu}-m\right\}\left|\Psi(r,t)\right\rangle=0\,, \tag{1.4}$$

where $\left\{\gamma^{\mu}\right\}_{\mu=0,1,2,3}$ are 4x4 matrices called Dirac gamma matrices taken such that $\left(\gamma^{0}\right)^{2}=-\left(\gamma^{1}\right)^{2}=-\left(\gamma^{2}\right)^{2}=-\left(\gamma^{3}\right)^{2}=1$ and $\left|\Psi(r,t)\right\rangle$ is a 4-component spinor wavefunction. More details can be found in [4]. Here we are interested in the 1+1 form of Dirac equation in general coupling environment.

In high energy physics (HEP) units are set by $\hbar=c=1$, Dirac equation in 1+1 dimensions with time-independent potentials takes the following most general form:

$$\left\{\gamma^{\mu}\left[i\partial_{\mu}-A_{\mu}(x)\right]-I_{2}S(x)-\gamma^{5}W(x)\right\}\left|\Psi(x,t\right\rangle=m\left|\Psi(x,t\right\rangle, \tag{1.5}$$

where m is the mass of the particle. $A_{\mu}=(V,U)$ is the vector potential (which is a 4-vector in 3+1 dimension), S(x) is the scalar potential and W(x) is the pseudo scalar potential. I_2 is the 2x2 unit matrix and $\{\gamma^\mu\}_{\mu=0,1}$ are the 2x2 Dirac gamma matrices defined below:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \rightarrow \gamma^5 = i\gamma^0\gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.6}$$

Thus, the wave equation (1.5) is now written in the following matrix form:

$$\begin{pmatrix} m + S(x) + V(x) & -\frac{d}{dx} + W(x) \\ \frac{d}{dx} + W(x) & -m - S(x) + V(x) \end{pmatrix} \begin{pmatrix} \psi^+(x) \\ \psi^-(x) \end{pmatrix} = \varepsilon \begin{pmatrix} \psi^+(x) \\ \psi^-(x) \end{pmatrix}, \tag{1.7}$$

where ε is the energy eigenvalue. More details could be found in Appendix B.1. Now, Eq. (1.7) expresses basically the problem that we are going to solve in this research with our method which is presented in the next section.

1.2 The Tridiagonal Matrix Representations Approach (TMRA)

There are certainly several exact and numerical techniques that have been used to solve the wave equation $H|\psi\rangle = E|\psi\rangle$, where H is the Hamiltonian and E is the energy eigenvalue which is either discrete (for bound states) or continuous (for scattering states) [3-9]. Our approach is a little bit different in the sense that we solve the eigenvalue equation indirectly without solving the differential equation as it is done usually [10, 11].

In some cases, especially when searching for algebraic or numerical solutions, we found that the wavefunction can be written in terms of square integrable basis (SIB) which is really the key point in this approach. So, we start by expanding the wavefunction in SIB as $\left|\psi(x,E)\right\rangle = \sum_m f_m(E)\left|\phi_m(x)\right\rangle$ where $\{\phi_m\}_{m=0}^{\infty}$ is a complete SIB set of functions of the variable x, i.e. they carry kinematic information only. On the other hand, $\{f_m\}_{m=0}^{\infty}$ is a set of expansion coefficients that depend on energy and potential parameters, which means that those coefficients carry all the physics about the relativistic problem. In this approach we define what we call the J-matrix in this basis defined as $J_{n,m} = \left\langle \phi_n \right| H - E \left| \phi_m \right\rangle$. We require this matrix to be tridiagonal and symmetric which transforms the wave equation into the following three-term recursion relation of the expansion coefficients of the wavefunction:

$$J_{n,n} f_n + J_{n,n-1} f_{n-1} + J_{n,n+1} f_{n+1} = 0 \tag{1.8}$$

where all the$J_{n,m}$'s are functions of energy and potential parameters. Solving this recursion relation we will be able to find the exact eigenstates together with the energy spectrum. Moreover, restricting the tridiagonal requirements on the wave operator helps to extend the class of solvable potentials within this space. This recursion relation is either compared to a well-known class of orthogonal polynomials or solved exactly using different approaches [12]. However, imposing the diagonal constraints on the wave operator J results in a conventional class of potentials. The continuous spectrum, for instance, is found by asymptotic analysis of the infinite sum over the basis. This is basically the summary of our approach, more details can be found in appendix B.

The choice of basis should be taken such that we achieve the tridiagonal requirements of the wave operator. In this work, we choose our basis elements to have the following form:

$$\phi_n(y) = A_n w_n(x) P_n(x), \tag{1.9}$$

Where A_n is just a normalization constant, $w_n(x)$ is some weight function which vanishes on the boundaries of the configuration space and $\{P_n(x)\}$ is a class of orthogonal polynomials.

We restrict ourselves here to two cases in which the configuration space is either finite or semi-infinite. For the finite space, we use Jacobi basis which is defined below:

$$\phi_n(y) = A_n (1-y)^\alpha (1+y)^\beta P_n^{(\mu,\nu)}(y), \tag{1.10}$$

Where $y \in [-1,1]$, $A_n = \sqrt{\frac{(2n+\mu+\nu+1)}{2^{\mu+\nu+1}} \frac{\Gamma(n+1)\Gamma(n+\mu+\nu+1)}{\Gamma(n+\nu+1)\Gamma(n+\mu+1)}}$ and $P_n^{(\mu,\nu)}(y)$ is Jacobi polynomial of order n. The parameters $\{\alpha,\beta,\mu,\nu\}$ will be determined later to satisfy the tridiagonal requirements.

The other basis is called Laguerre basis which is suitable for systems (Hamiltonians) that are defined on a semi-infinite space. The general form of this basis is given below:

$$\phi_n(y) = A_n y^\alpha e^{-\beta y} L_n^\nu(y), \tag{1.11}$$

Where $y \in [0,\infty)$, $A_n = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\nu+1)}}$ and $\{\alpha,\beta,\nu\}$ are constants to be determined later.

In the next chapter we will apply the tridiagonal matrix technique to Dirac equation given in (1.7) and calculate the J-matrix in a general basis. Then, we will consider the two special basis sets, Jacobi and Laguerre bases, write the J-matrix in both bases and introduce the compact form of the tridiagonal symmetric matrix J together with the constrains on the parameters in the problem.

CHAPTER 2

General Formulation of the Problem

This chapter gives the mathematical approach to the problem and is divided into three main sections. The first section gives a general formulation of Dirac equation in general square integrable basis (SIB) set and we will show the derivations of the general form of the J-matrix. The next two sections talks about the wave operator formulation in Jacobi and Laguerre bases. We also give a general procedure on how we extract out the solvable potentials within the space in which the J-matrix becomes tridiagonal and symmetric. Most of the mathematical details are left to the reader in the appendices.

2.1 Dirac Equation and the General from of Dirac Operator

The J-matrix in a general SIB set $\{\phi_m\}_{m=0}^{\infty}$ is defined below:

$$J_{n,m} = \langle \phi_n | (H-\varepsilon) | \phi_m \rangle, \tag{2.1}$$

where H is the Hamiltonian in Eq. (1.7) and the basis spinor written in component form as $|\phi_n\rangle = (\phi_n^+, \phi_n^-)^T$. Using Eq. (1.7), we can write the wave operator elements as:

$$J_{n,m} = \langle \phi_n^+ | Q_+ | \phi_m^+ \rangle + \langle \phi_n^- | Q_- | \phi_m^- \rangle + \langle \phi_n^+ | -\frac{d}{dx} + W(x) | \phi_m^- \rangle + \langle \phi_n^- | \frac{d}{dx} + W(x) | \phi_m^+ \rangle, \tag{2.2}$$

where $Q_{\pm} = V(x) \pm S(x) \pm m - \varepsilon$.

In Dirac equation we can write the spinor lower component in terms of the upper component as shown below:

$$\psi^{-}(x) = \frac{1}{m + S(x) + \varepsilon - V(x)} \left\{ \frac{d}{dx} + W(x) \right\} \psi^{+}(x), \tag{2.3}$$

The singularities of the term $\frac{1}{m + S(x) + \varepsilon - V(x)}$ do create what we call *spurious modes* or *spurious eigenvalues* of Dirac operator as discussed in [13]. One useful way to avoid such modes is to use what is called *the kinetic balance relation* (KBR) as discussed in chapter 5 in [14], which basically relates the spinor components as follows:

$$\psi^{-}(x) \propto W(x) + \frac{d}{dx} \psi^{+}(x), \tag{2.4}$$

The basic idea in the KBR is that we choose a basis functions $\left\{ \phi_n^{+}(x) \right\}_{n=0}^{\infty}$ to expand the upper component of the spinor wavefunction and then we use the KBR to calculate the basis functions for lower component of the spinor wavefunction $\left\{ \phi_n^{-}(x) \right\}_{n=0}^{\infty}$. For our case, the KBR reads as follows:

$$\phi_n^{-}(x) = \frac{1}{\lambda} \left[\frac{d}{dx} + R(x) \right] \phi_n^{+}(x) = \frac{1}{\lambda} \left[y' \frac{d}{dy} + R(y) \right] \phi_n^{+}(y), \tag{2.5}$$

where λ is a scale factor. The transformation $x \to y(x)$ is made such that we transform the given space to the domain of the selected basis, in other words, to make the space of the basis compatible with the domain of the Hamiltonian.

Here we are interested in the finite space where $y \in [-1, 1]$ (Jacobi basis), and the semi-infinite space where $y \in [0, \infty)$ (Laguerre basis). Using Eq. (2.5) in Eq. (2.2), with few integration by parts given in Appendix B.2, we can deduce the following form of the wave operator in a general basis:

$$\begin{aligned} J_{n,m} &= \frac{-1}{\lambda} \left\langle \phi_n^+ \right| \left\{ \left(2 + \lambda^{-1} Q_-\right) \left[y'^2 \frac{d^2}{dy^2} + y'' \frac{d}{dy} \right] + \lambda^{-1} y' V_-' \frac{d}{dy} \right\} \left| \phi_m^+ \right\rangle \\ &+ \frac{1}{\lambda} \left\langle \phi_n^+ \right| \left[\lambda^{-1} Q_- \left(R^2 - R'\right) + R\left(2W - \lambda^{-1} V_-'\right) - R' - W' + \lambda Q_+ \right] \left| \phi_m^+ \right\rangle \end{aligned}, \tag{2.6}$$

where the dot over the variable stands for the derivative with respect to y. Our goal now is to use this formula of the J-matrix for a specific choice of basis and make it tridiagonal and symmetric as shown in the next sections.

2.2 The J-matrix in Jacobi Basis

We expand the upper spinor component in terms of Jacobi basis:

$$\phi_n^+(y) = A_n (1-y)^\alpha (1+y)^\beta P_n^{(\mu,\nu)}(y), \tag{2.7}$$

where $A_n = \sqrt{\frac{(2n+\mu+\nu+1)}{2^{\mu+\nu+1}} \frac{\Gamma(n+1)\Gamma(n+\mu+\nu+1)}{\Gamma(n+\nu+1)\Gamma(n+\mu+1)}}$ is just a normalization constant, $y \in [-1, 1]$ and $P_n^{(\mu,\nu)}(y)$ is Jacobi polynomial of order n. The parameters $\{\alpha, \beta, \mu, \nu\}$ will be constrained in order to satisfy the tridiagonal requirements.

Starting with Eq. (2.7) with few steps of algebra and using the properties of the Jacobi polynomials (see Appendix B.3), we can write the wave operator in Jacobi basis in the following compact form:

$$J_{n,m} = 2c\lambda\,\Omega_{n,m} + 2\lambda A_n A_m \int_{-1}^{+1} (1-y)^{\mu}(1+y)^{\nu} G(y) P_n^{(\mu,\nu)}(y) P_m^{(\mu,\nu)}(y) dy\,, \tag{2.8}$$

where,

$$\begin{aligned} G(y) = \frac{\lambda Q_+}{2}\frac{1-y^2}{y'^2} - c\left[\beta - \alpha - (\alpha+\beta)y + \frac{1-y^2}{y'}R\right] + \left(1+\frac{Q_-}{2\lambda}\right)\left\{\left(n+\frac{\mu+\nu+1}{2}\right)^2 - \frac{1}{4}\left(\mu^2+\nu^2+2ab-1\right)\right. \\ \left. -\frac{1}{4}\left[\mu^2-(a-1)^2\right]\frac{1+y}{1-y} - \frac{1}{4}\left[\nu^2-(b-1)^2\right]\frac{1-y}{1+y} + \frac{1-y^2}{y'^2}\left(R^2 - y'\dot{R}\right)\right\} \end{aligned}, \tag{2.9}$$

and,

$$\begin{aligned} \Omega_{n,m} = (\nu-\mu)\frac{2n(n+\mu+\nu+1)}{(2n+\mu+\nu)(2n+\mu+\nu+2)}\delta_{n,m} - (\mu+\nu+2)\left[\frac{1}{2n+\mu+\nu}\sqrt{\frac{n(n+\mu)(n+\nu)(n+\mu+\nu)}{(2n+\mu+\nu-1)(2n+\mu+\nu+1)}}\,\delta_{n,m+1}\right. \\ \left. + \frac{1}{2n+\mu+\nu+2}\sqrt{\frac{(n+1)(n+\mu+1)(n+\nu+1)(n+\mu+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+3)}}\,\delta_{n,m-1}\right] \end{aligned}, \tag{2.10}$$

The tridiagonal constraints in Jacobi basis are summarized below (see Appendix B.3):

$$\begin{aligned} &2\alpha + a = \mu + 1 \\ &2\beta + b = \nu + 1 \\ &R - W + y'\frac{\dot{Q}_-}{2\lambda} = cy' \\ &G(y) = \rho y + \sigma \end{aligned}, \tag{2.11}$$

Using the last line in equation (2.11), we can write (2.11) in the following compact form:

$$J_{n,m} = 2\lambda \left\{ \begin{array}{l} \left[\sigma + \dfrac{\rho\left(\nu^2-\mu^2\right)+2nc(\nu-\mu)(n+\mu+\nu+1)}{(2n+\mu+\nu)(2n+\mu+\nu+2)} \right] \delta_{n,m} \\ + \left[\begin{array}{l} \dfrac{2\rho - c(\mu+\nu+2)}{(2n+\mu+\nu+2)} \sqrt{\dfrac{(n+1)(n+\mu+\nu+1)(n+\mu+1)(n+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+3)}} \delta_{n,m-1} \\ + \dfrac{2\rho - c(\mu+\nu+2)}{(2n+\mu+\nu)} \sqrt{\dfrac{n(n+\mu+\nu)(n+\nu)(n+\mu)}{(2n+\mu+\nu-1)(2n+\mu+\nu+1)}} \delta_{n,m+1} \end{array} \right] \end{array} \right\}, \tag{2.12}$$

The basic procedure here is that we try to linearize (2.12) in y such that we make the J-matrix tridiagonal and symmetric. The general procedure that we are going to follow in this basis is summarized below:

- Take some special cases of the potentials. For example, by taking S=V, W=0.
- Consider a specific choice of transformation by picking some values of a and b in the transformation equation $y' = \lambda(1-y)^a(1+y)^b$ to ensure the compatibility with the basis prefactor in (2.7).
- Using the constraints in Eq. (2.11) together with the previous steps we find the corresponding potentials that make Eq. (2.9) linear in y.
- For each set of potentials we construct the J-matrix (2.12) together with the associated three-term recursion relation.
- We try to solve the recursion relation either by comparison with well-known class of orthogonal polynomials or by solving it exactly using different techniques.
- The potentials, the eigenstates, and the energy spectrum will also be plotted using Mathematica.
- Finally, we study the bound states as well as the scattering states using Schrodinger picture, i.e. by decoupling Dirac equation into two Schrodinger equations.

2.3 J-matrix in Laguerre Basis

Laguerre basis is defined below:

$$\phi_n(y) = A_n y^{\alpha} e^{-\beta y} L_n^{\nu}(y), \tag{2.13}$$

where $A_n = \sqrt{\dfrac{\Gamma(n+1)}{\Gamma(n+\nu+1)}}$, $y \in [0,\infty)$, and $L_n^{\nu}(y)$ is the nth order Laguerre polynomial. $\{\alpha,\beta,\nu\}$ are constant parameters to be restricted by the tridiagonal constraints.

Finding the first two derivatives of Eq. (2.13) and considering the recursion relations of the Laguerre polynomials, we can write down the following form of the J-matrix in this basis:

$$\begin{aligned} J_{n,m} &= c\lambda\left[-2n\delta_{n,m} + \sqrt{n(n+\nu)}\delta_{n,m+1} + \sqrt{(n+1)(n+\nu+1)}\delta_{n,m-1}\right] \\ &\quad + 2\lambda A_n A_m \int_0^{\infty} y^{\nu} e^{-y} G(y) L_n^{\nu} L_m^{\nu} dy \end{aligned}, \tag{2.14}$$

where,

$$\begin{aligned} G(y) &= \frac{\lambda Q_+}{2}\frac{y}{y'^2} - c\left(\alpha - \beta y + \frac{y}{y'}R\right) + \left(1 + \frac{Q_-}{2\lambda}\right)\times \\ &\left[\frac{y}{y'^2}\left(R^2 - y'\dot{R}\right) + \left(n + \frac{\nu - ab + 1}{2}\right) - \frac{1}{4}\frac{\nu^2 - (a-1)^2}{y} + \frac{b^2 - 1}{4}y\right] \end{aligned}, \tag{2.15}$$

The tridiagonal constraints here are written bellow (See appendix B.4):

$$\begin{aligned} &2\alpha + a = \nu + 1 \\ &2\beta - b = 1 \\ &R - W + y'\frac{\dot{Q}_-}{2\lambda} = cy' \\ &G(y) = \rho y + \sigma \end{aligned}, \tag{2.16}$$

Using the last line in equation (2.14) into Eq. (2.12), we can write the J-matrix in the following closed form:

$$J_{n,m} = 2\lambda\left\{\left[\sigma + 2\left(n + \frac{\nu+1}{2}\right)\rho - cn\right]\delta_{n,m} + \left(\frac{c}{2} - \rho\right)\left[\sqrt{n(n+\nu)}\delta_{n,m+1} + \sqrt{(n+1)(n+\nu+1)}\delta_{n,m-1}\right]\right\}, \tag{2.17}$$

The general procedure in this basis is similar to that mentioned for Jacobi basis but with coordinate transformation taken such that $\frac{dy}{dx} = \lambda y^a e^{by}$.

In the summary of this chapter, the main goal here is to extend the class of solvable potentials within the tridiagonal space of the Dirac operator. Once we have this set of potentials $\{S, V, W\}$, we can construct the J-matrix that is associated with them for each basis set. The next step will be that we use this specific form of the wave operator to solve the three-term recursion relation given in Eq. (1.7). This will give the full details about the eigenstates and the corresponding energy spectrum. The following diagram summarizes all these steps.

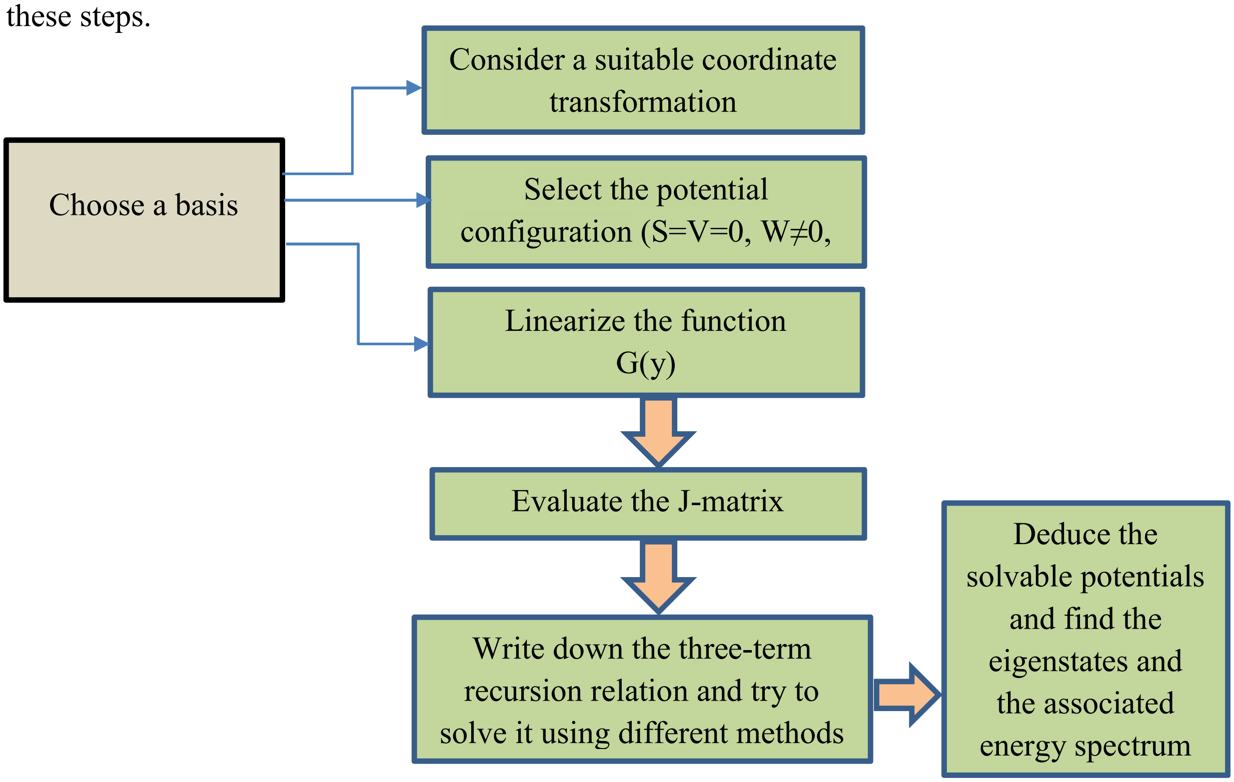

Figure 1: Flow chart of the general procedure

CHAPTER 3

RESULTS AND DISCUSIONS

This chapter is divided into two main sections. The first section shows the class of solvable potentials we obtained in Laguerre basis. The next section discusses the set of potential functions that have been solved within Jacobi basis. The potential functions, the energy spectrum and the eigenstates have been plotted for each set of potentials.

3.1 Solvable potentials within Laguerre Basis

In this section we will show the class of potentials that have been obtained within this basis. First of all, we consider the simplest case in which S=V=0 and W is nonzero together with the coordinate transformation $y\left(x\right)=\frac{\left(\lambda x\right)^2}{4}$ makes Eq. (2.15) take the following form:

$$G(y)=\frac{m-\varepsilon}{2\lambda}-c\left(\alpha-\beta y+\frac{\sqrt{y}}{\lambda}R\right)+\left(1-\frac{m+\varepsilon}{2\lambda}\right)\times\left[\frac{1}{\lambda^2}\left(R^2-\lambda\sqrt{y}\,\dot{R}\right)+\left(n+\frac{\nu+1}{2}\right)-\frac{1}{4}\frac{\nu^2-\frac{1}{4}}{y}-\frac{1}{4}y\right],$$

(3.1.1)

To make (3.1.1) linear in y one would take c=0, and require the following condition:

$$\frac{1}{\lambda^2}\left(R^2-\lambda\sqrt{y}\,\dot{R}\right)=\frac{1}{4}\frac{\nu^2-\frac{1}{4}}{y}, \tag{3.1.2}$$

The general solution of (3.1.2) is given below:

$$R(y)=\frac{\lambda}{4\sqrt{y}}\left(-(1+2\nu)+\frac{4\nu\delta}{y^{\nu}+\delta}\right), \tag{3.1.3}$$

Where we choose $\delta \geq 0$ and ν is taken to be positive even integer to avoid having singularities of the potential function in the domain $y \in [0,\infty[$. The third line in Eq. (2.16) suggests that the solution for the pseudopotential is identical to R(y). Thus, the pseudopotential as a function of x reads as:

$$W(x)=\frac{1}{2x}\left(-(1+2\nu)+\frac{4^{\nu+1}\nu\delta}{(\lambda x)^{2\nu}+4^{\nu}\delta}\right), \tag{3.1.4}$$

The plot of W(x) for $\{\delta=0, \nu=2\}$ and $\{\delta=1, \nu=6\}$, where we took $\lambda=1$, is shown below:

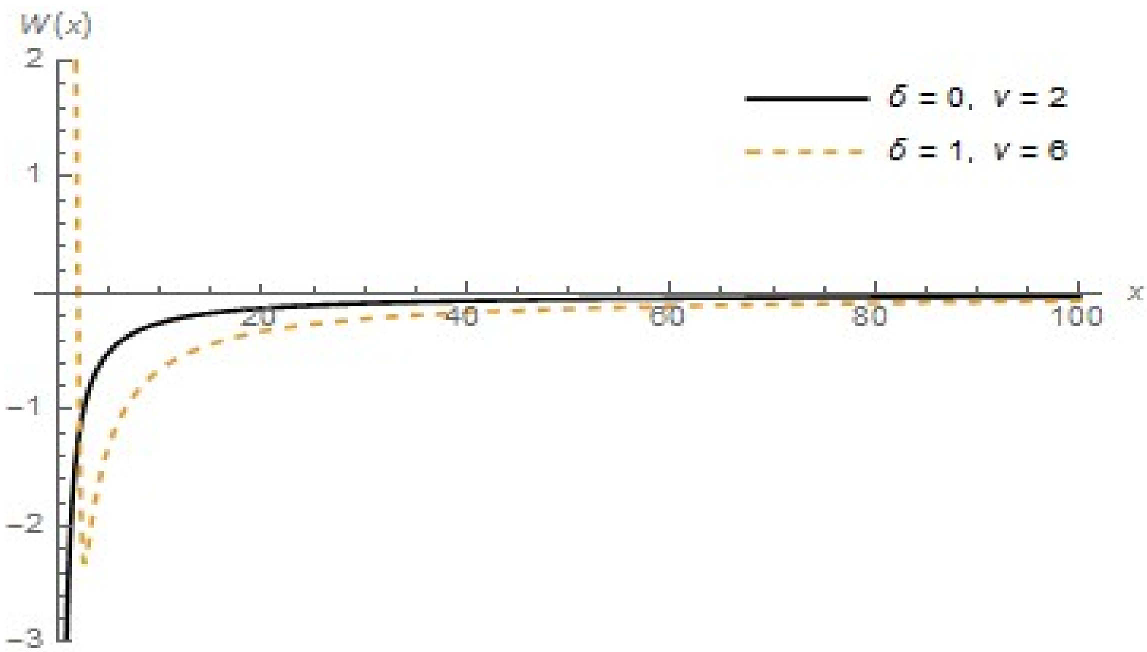

Figure 2: Plot of W(x) for two choices of parameters $\{\delta=0, \nu=2\}$ (black continuous curve) and $\{\delta=1, \nu=6\}$ (dashed curve) where $\lambda=1$.

Now, the function G(y) associated with this potential can be written in the following form:

$$G(y)=-\frac{1}{4}\left(1-\frac{m+\varepsilon}{2\lambda}\right)y+\frac{m-\varepsilon}{2\lambda}+\left(1-\frac{m+\varepsilon}{2\lambda}\right)\left(n+\frac{\nu+1}{2}\right), \tag{3.1.5}$$

Using Eq. (2.17) we can write the corresponding J-matrix as:

$$J_{n,m}=2\lambda\left\{\begin{array}{c}\left[\frac{m-\varepsilon}{2\lambda}+\frac{1}{2}\left(n+\frac{\nu+1}{2}\right)\left(1-\frac{m+\varepsilon}{2\lambda}\right)\right]\delta_{n,m}\\+\frac{1}{4}\left(1-\frac{m+\varepsilon}{2\lambda}\right)\left[\sqrt{n(n+\nu)}\delta_{n,m+1}+\sqrt{(n+1)(n+\nu+1)}\delta_{n,m-1}\right]\end{array}\right\}, \quad (3.1.6)$$

The associated three-term recursion relation becomes:

$$zf_n=\left(n+\frac{\nu+5}{2}\right)f_n+\frac{1}{2}\left[\sqrt{n(n+\nu)}f_{n-1}+\sqrt{(n+1)(n+\nu+1)}f_{n+1}\right], \quad (3.1.7)$$

where: $z=2\left[1-\frac{m}{\lambda}\right]/\left(1-\frac{m+\varepsilon}{2\lambda}\right)$.

The plot the four lowest energy eigenvalues versus ν for m=1, λ=1.5:

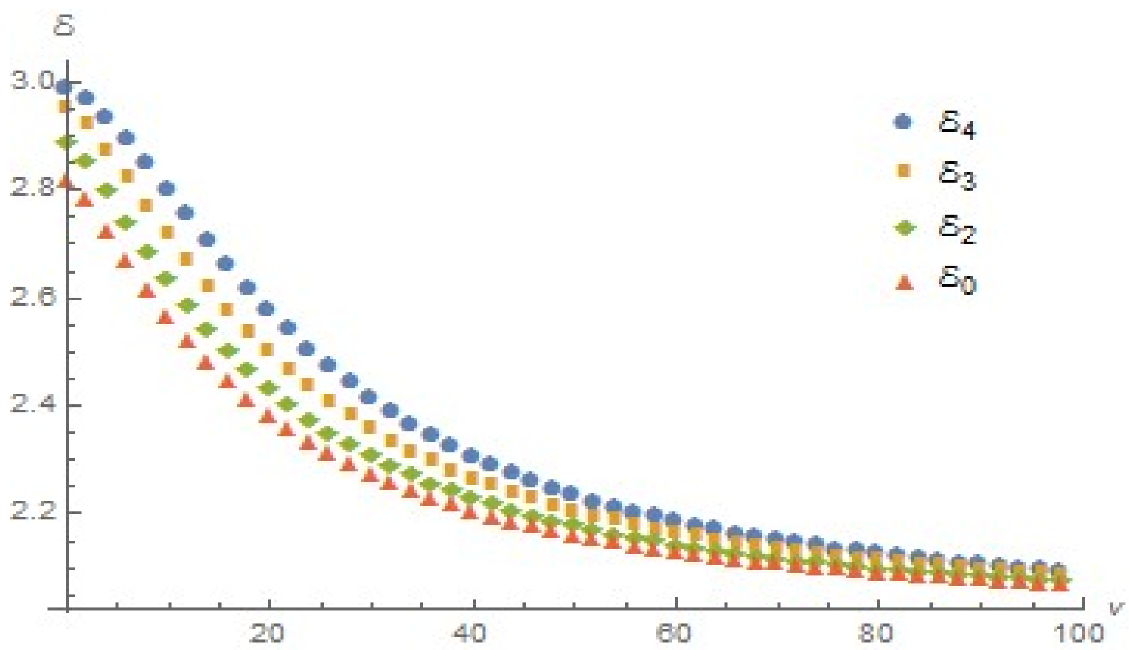

Figure 3: The four lowest energy eigenvalues versus the potential parameter v where we took m=1, λ=1.5. It should be noticed here the curves are almost coincide (isospectral).

The recursion relation in (3.1.7) is still under study wether it can be related to well-known class of orthogonal polynomials or it leads to a new class of orthogonal polynomials. For the moment, we can generate these functions for any specific values of the potential parameters. As an example, we show some of these polynomials for ν=2 in the table below:

Table 1: Four polynomial solutions of (3.1.7) where we took $f_{-1}(z)=0, f_0(z)=1$ and the variable z is related to the energy by $z=2\left[1-\frac{m}{\lambda}\right]/\left(1-\frac{m+\varepsilon}{2\lambda}\right)$.

n	$f_n(z)$
1	$-\frac{13}{3}+\frac{2z}{3}$
2	$\frac{31}{\sqrt{5}}-\frac{28z}{3\sqrt{5}}+\frac{2z^2}{3\sqrt{5}}$
3	$\frac{26\sqrt{\frac{5}{33}}}{3}-\frac{527}{\sqrt{165}}-\frac{4}{3}\sqrt{\frac{5}{33}}z+\frac{662z}{3\sqrt{165}}-2\sqrt{\frac{15}{11}}z^2+\frac{4z^3}{3\sqrt{165}}$
4	$\frac{6125+2z(-1772+z(366+(-32+z)z))}{9\sqrt{55}}$

The following graph shows the behavior of the un-normalized spinor wavefunction for the ground state for different values of ν. As shown in the plots we conclude that the spinor components are almost $\frac{\pi}{2}$ out of phase. The blue curve represents the upper component while the yellow curve represents the lower component of the spinor wavefunction.

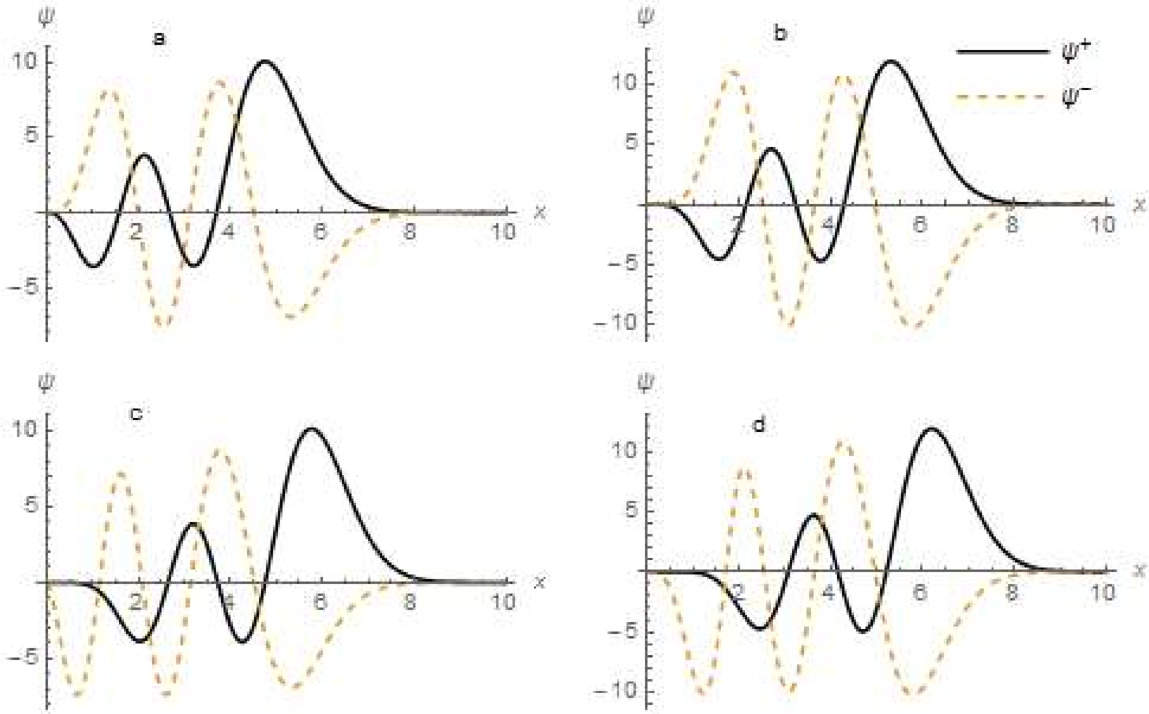

Figure 4: The plots show the oscillatory behavior of the spinor components for different values of ν and δ for the ground state. (a, b) for $\delta=0$ and $\nu=2,4$ and (c, d) for $\delta=1$ and $\nu=2,4$. Here we took m= 1, $\lambda=1.5$.

Another way to make (3.1.1) linear in y is by considering the following particular solution of R:

$$R(y)=\lambda\left(\gamma\sqrt{y}+\frac{\tau}{\sqrt{y}}\right), \tag{3.1.8}$$

Where γ and τ are constant parameters.

The following particular solution of W is suggested by the third line in Eq. (2.14):

$$W(y) = \lambda\left(\left(\gamma - c\right)\sqrt{y} + \frac{\tau}{\sqrt{y}}\right), \tag{3.1.9}$$

The plot of (3.1.9) for c=0, λ=1.5 and different values of γ and τ is shown below:

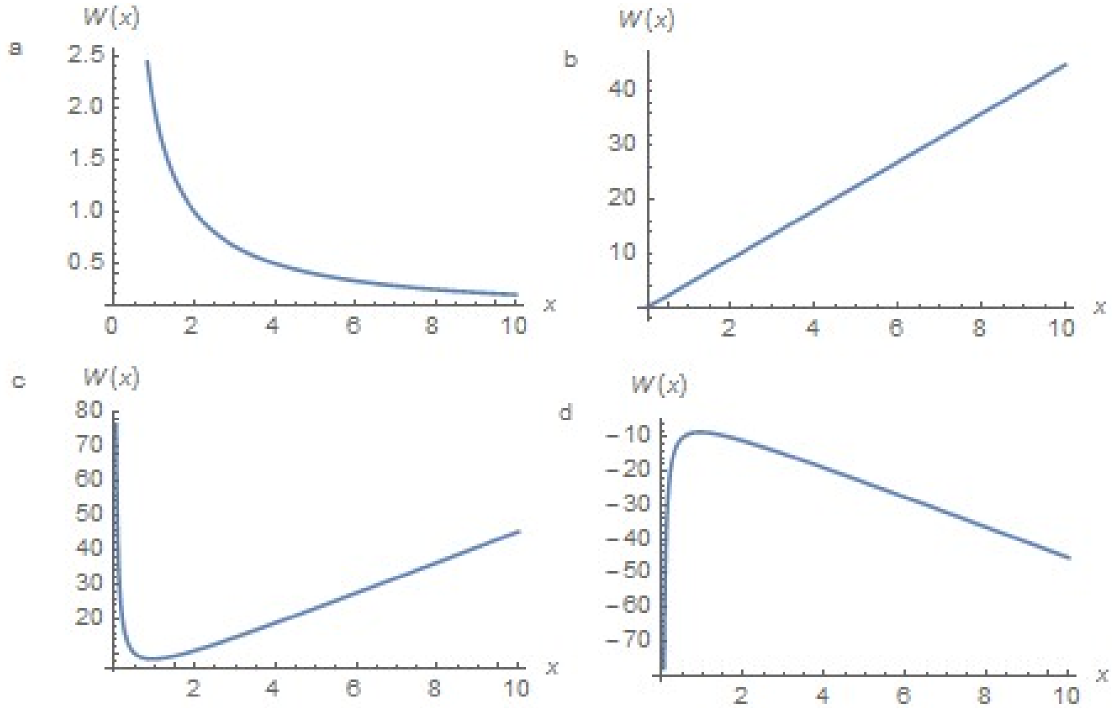

Figure 5: The plot of W(x) vs x. Here we took c=0, λ=1.5. (a) The blue curve for γ=0, τ=1. (b) The red curve (line) when τ=0, γ=4. (c) The green curve for γ=4, τ=2. (d) The black curve is for γ= -4, τ= -2.

By substituting Eq. (3.1.9) back in Eq. (3.1.1), we impose the following constraint to make

G(y) linear: $\left(\tau + \frac{1}{4}\right)^2 = \frac{1}{4}\nu^2$, (3.1.10)

Thus, in this case Eq. (3.1.1) reduces to the following form:

$$G(y)=\left[\left(1-\frac{m+\varepsilon}{2\lambda}\right)\left(\gamma^2-\frac{1}{4}\right)-c\left(\gamma-\frac{1}{2}\right)\right]y+\frac{m}{\lambda}-1-c\left(\frac{2\nu+1}{4}+\tau\right)+\left(1-\frac{m+\varepsilon}{2\lambda}\right)\times \\ \left[\left(n+\frac{\nu+3}{2}\right)+2\gamma\left(\tau-\frac{1}{4}\right)\right], \tag{3.1.11}$$

The associated J-matrix for c=0 reads as follows:

$$J_{n,m}=2\lambda\left\{\begin{array}{l}\left[\frac{m}{\lambda}-1+\gamma\left(2\tau-\frac{1}{2}\right)\left(1-\frac{m+\varepsilon}{2\lambda}\right)+\left(n+\frac{\nu+1}{2}\right)\left(1-\frac{m+\varepsilon}{2\lambda}\right)\left(2\gamma^2+\frac{1}{2}\right)\right]\delta_{n,m} \\ -\left(1-\frac{m+\varepsilon}{2\lambda}\right)\left(\gamma^2-\frac{1}{4}\right)\left[\sqrt{n(n+\nu)}\delta_{n,m+1}+\sqrt{(n+1)(n+\nu+1)}\delta_{n,m-1}\right]\end{array}\right\}, \tag{3.1.12}$$

The three-term recursion relation corresponding to (3.1.12) is given below:

$$zf_n=-\left(n+\frac{\nu+1}{2}\right)\left(2\gamma^2+\frac{1}{2}\right)f_n \\ +\left(\gamma^2-\frac{1}{4}\right)\left[\sqrt{n(n+\nu)}f_{n-1}+\sqrt{(n+1)(n+\nu+1)}f_{n+1}\right], \tag{3.1.13}$$

Where: $z=\left[\frac{\frac{m}{\lambda}-1}{\left(1-\frac{m+\varepsilon}{2\lambda}\right)}+\gamma\left(2\tau-\frac{1}{2}\right)\right]$.

Making the following transformations:

$\cosh\theta=\frac{\left(2\gamma^2+\frac{1}{2}\right)}{\left(2\gamma^2-\frac{1}{2}\right)}$ and $iy=\frac{\frac{m}{\lambda}-1}{2\gamma\left(1-\frac{m+\varepsilon}{2\lambda}\right)}+\left(\tau-\frac{1}{4}\right)$, we can write the solutions of (3.1.13) in terms of the discrete Miexner polynomials as (See Appendix A4):

$$f_n(k;\theta)=\left(1-e^{-2\theta}\right)^{\frac{\nu+1}{2}}e^{-k\theta}\sqrt{\frac{\Gamma(k+\nu+1)}{\Gamma(\nu+1)\Gamma(k+1)}}M_n^{\nu+1}\left(k;e^{-2\theta}\right), \tag{3.1.14}$$

where k is an integer that stands for the k[th] bound state.

The energy spectrum can be calculated using the infinite spectrum formula associated with these polynomials:

$$y^2 = \left(n + \frac{\nu+1}{2}\right)^2 \rightarrow \varepsilon_n^{\pm} = 2\lambda - m + \frac{(\lambda - m)}{\gamma\left[\left(\frac{1}{4} - \tau\right) \pm \left(n + \frac{\nu+1}{2}\right)\right]}, \tag{3.1.15}$$

It is clear from Eq. (3.1.15) that this system has only bound states. Now, we plot the four lowest energy eigenvalues versus a range of values of the parameter γ for m=ν=1, λ=1.5 and τ=0.25 for ε_n^+ and ε_n^- in the following figure:

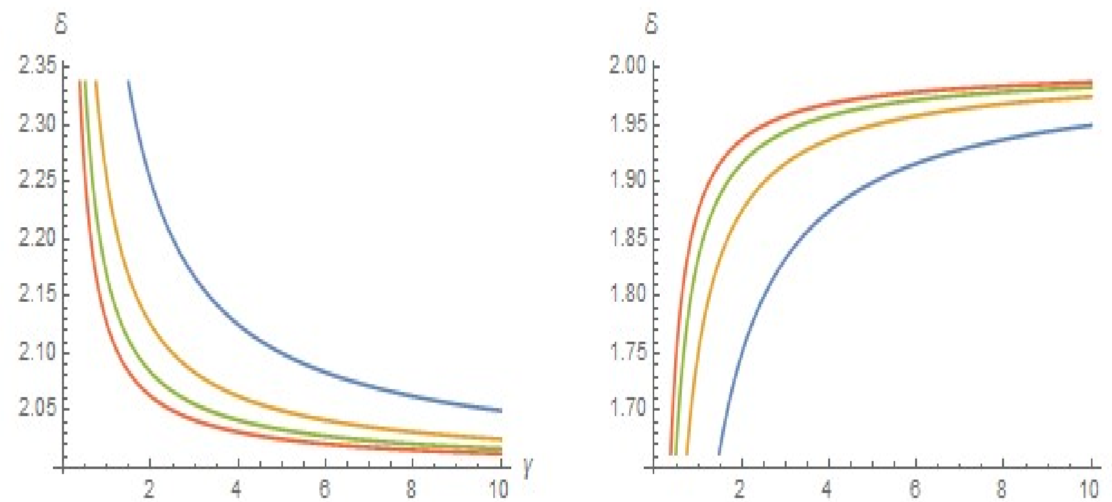

Figure 6: Plot of the energy spectrum for m=ν=1, λ=1.5 and τ=0.25. On the right for ε_n^- and on the left for ε_n^+.

Additionally, we plotted the four lowest energy eigenvalues numerically to compare our results as shown in the next figure.

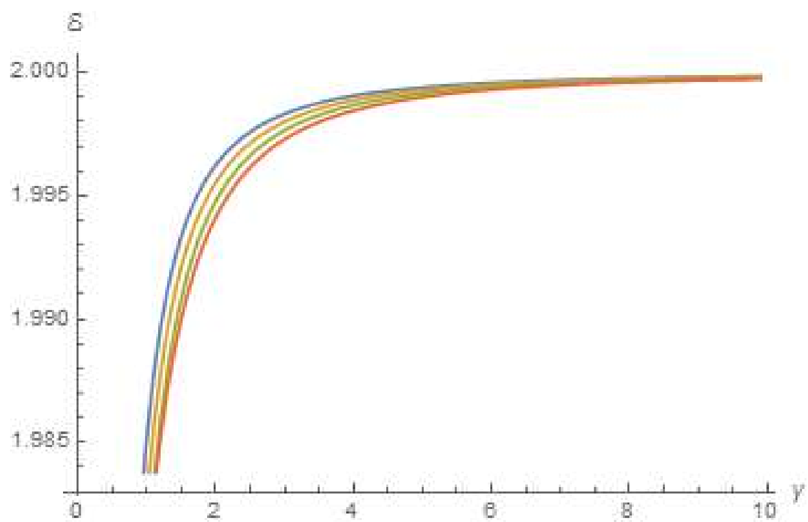

Figure 7: Plot of the energy spectrum numerically for m=ν=1, λ=1.5 and τ=0.25. Comparing this with the right hand side of figure (6) we see similar behavior of the spectrum.

We plot the two lowest bound states spinor wavefunction components as follows:

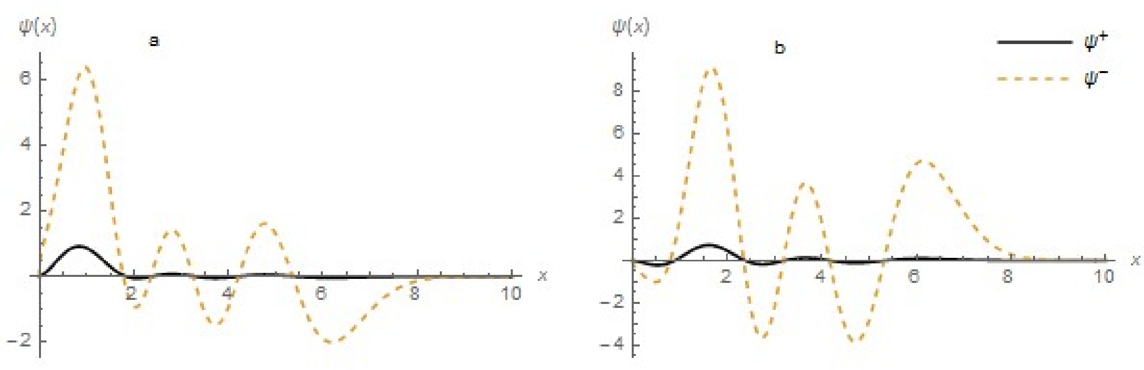

Figure 8: Plot of the spinor components for the first two lowest states. We took λ=1.5, m=ν=1 and τ=0.25. a) for the ground state and b) for the 1st excited state.

The last solvable potential we mention here for this case, S=V=0, is obtained by taking c=0, ν=1 and choosing R to satisfy the following differential equation:

$$\frac{1}{\lambda^2}\left(R^2-\lambda\sqrt{y}\dot{R}\right)-\frac{3}{16y}=\frac{k^2}{4}y, \tag{3.1.16}$$

Where k is just an integer which is chosen later to be different from ±1 and 0.

The general solution to (3.1.16) is given below:

$$R(y)=\frac{\lambda}{4\sqrt{y}}\left\{\frac{(1-2ky)e^{ky}+\delta\left(k+2k^2y\right)}{e^{ky}+k\delta}\right\}, \tag{3.1.17}$$

Where δ is a real constant chosen such that kδ is non-negative.

The associated pseudopotential as a function of x is given by:

$$W(x)=\frac{1}{2x}\left\{\frac{\left(1-k\frac{(\lambda x)^2}{2}\right)e^{k\frac{(\lambda x)^2}{4}}+k\delta\left(1+k\frac{(\lambda x)^2}{2}\right)}{e^{k\frac{(\lambda x)^2}{4}}+k\delta}\right\}, \tag{3.1.18}$$

The function G(y) associated with this potential reads as follows:

$$G(y)=\frac{1}{4}\left(k^2-1\right)\left(1-\frac{m+\varepsilon}{2\lambda}\right)y+\frac{m-\varepsilon}{2\lambda}+\left(1-\frac{m+\varepsilon}{2\lambda}\right)(n+1), \tag{3.1.19}$$

Thus, we can write the J-matrix for this case as:

$$J_{n,m}=2\lambda\left\{\begin{array}{c}\left[\frac{m-\varepsilon}{2\lambda}+\frac{1}{2}\left(k^2+1\right)(n+1)\left(1-\frac{m+\varepsilon}{2\lambda}\right)\right]\delta_{n,m}\\ -\frac{1}{4}\left(k^2-1\right)\left(1-\frac{m+\varepsilon}{2\lambda}\right)\left[\sqrt{n(n+\nu)}\delta_{n,m+1}+\sqrt{(n+1)(n+\nu+1)}\delta_{n,m-1}\right]\end{array}\right\}, \tag{3.1.20}$$

The plot of W(x) for different choices of parameters is shown the figure below:

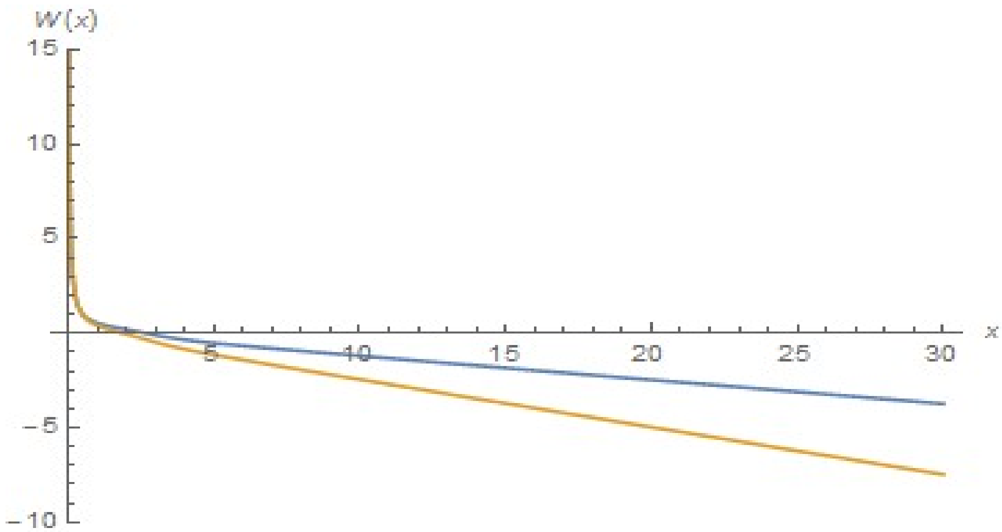

Figure 9: The plot of W(x) for {δ=k=1} and {δ=k=-1}.

It should be notices here that taking δ=0 gives solutions identical to (3.1.9). Now, the three-term recursion relation associated with this J-matrix is given below:

$$\left[\frac{m-\varepsilon}{2\lambda}\right]f_n = -\frac{1}{2}\left(k^2+1\right)\left(n+1\right)\left(1-\frac{m+\varepsilon}{2\lambda}\right)f_n + \frac{1}{4}\left(k^2-1\right)\left(1-\frac{m+\varepsilon}{2\lambda}\right)\left[\sqrt{n(n+1)}f_{n-1}+\sqrt{(n+1)(n+2)}f_{n+1}\right], \qquad (3.1.21)$$

Making the following transformations:

$\cosh\theta = \dfrac{k^2+1}{k^2-1}$, and $iz = \dfrac{\left[\frac{m-\varepsilon}{\lambda}\right]}{2k\left(1-\frac{m+\varepsilon}{2\lambda}\right)}$ we transform (3.1.21) to the following relation:

$$\left[iz\sinh\theta\right]f_n = -\left[\left(n+1\right)\cosh\theta\right]f_n + \frac{1}{2}\left[\sqrt{n(n+1)}f_{n-1}+\sqrt{(n+1)(n+2)}f_{n+1}\right], \qquad (3.1.22)$$

Again, solutions to (3.1.22) are the discrete Meixner polynomials which are given by:

$$f_n(s;\theta) = \left(1-e^{-2\theta}\right)e^{-s\theta}\sqrt{\frac{\Gamma(s+2)}{\Gamma(s+1)}}M_n^2\left(s;e^{-2\theta}\right), \tag{3.1.23}$$

Where s stands for the s-bound states. The energy spectrum is now written as:

$$\varepsilon_n^{\pm} = \frac{m \pm k(n+1)(m-2\lambda)}{\left(\mp k(n+1)+1\right)}, \tag{3.1.24}$$

We plot the energy spectrums versus the range of k-values for m=1, λ=1.5 as follows:

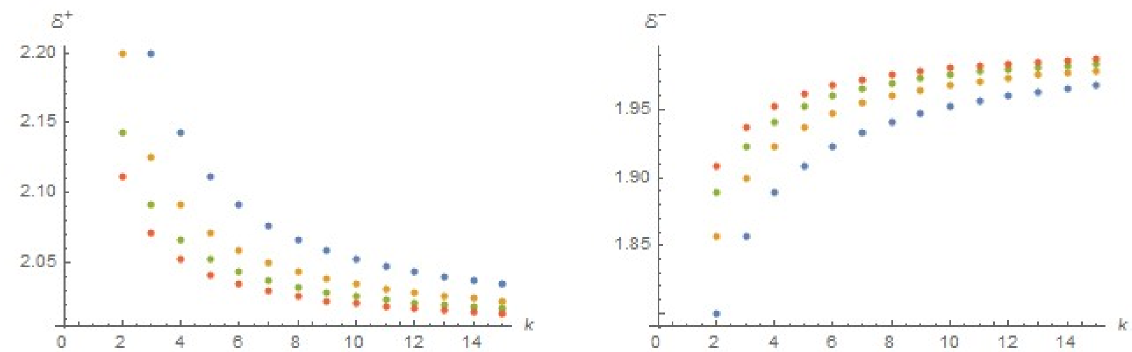

Figure 10: Plot of the energy spectrum versus k for m=1, λ=1.5. Notice here k is an integer, which is different from ±1, to avoid having energy blow up in the plus branch.

The s-state spinor wavefunction components are written as follows:

$$\psi_s^{\pm}(x,\varepsilon_s) = \left(1-e^{-2\theta}\right)e^{-s\theta}\sqrt{\frac{\Gamma(s+2)}{\Gamma(s+1)}}\sum_n M_n^2\left(s;e^{-2\theta}\right)\phi_n^{\pm}(x), \tag{3.1.25}$$

where the basis components are related to each other through the kinetic balance relation with the function R given in (3.1.17) for ν=1 and β=0.5.

It should be noticed here that the potentials mentioned above are not just solutions to Dirac equation for the pseudopotential but also new solutions to Schrodinger equation for the isospectral potentials $W^2(x)+\frac{dW}{dx}$ and $W^2(x)-\frac{dW}{dx}$ as shown in appendix C. In the next pages we will show some other results that belong to the case in which the spinor particle having spin symmetric coupling, that is, when S=V and we considered W either zero or nonzero.

The following figure shows the plot of spinor wavefunction components for both the ground state and the 1st excited state for some choice of parameters indicated in the figure:

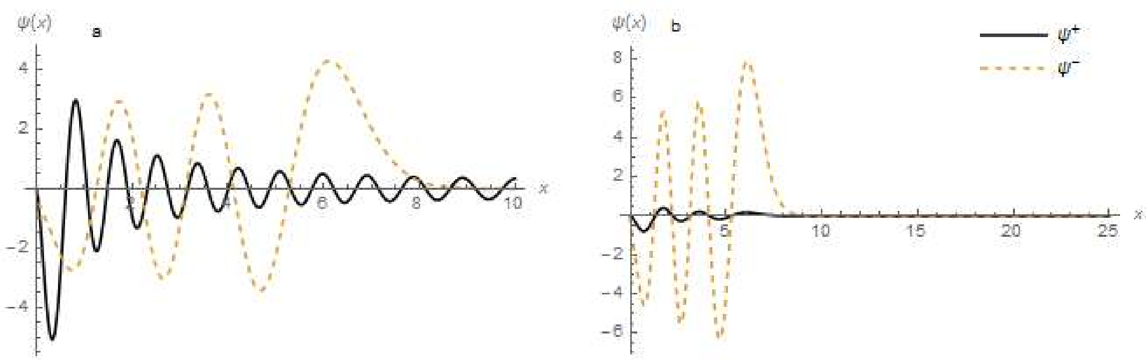

Figure 11: Plot of the spinor wavefunction components for the ground state (a) and the first excited state (b) for k=2, ν=m=δ=1, and λ=1.5.

For the case where S=V and W is nonzero, the function G(y) takes the following form:

$$G(y)=\frac{2V+m-\varepsilon}{2\lambda}-c\left(\alpha-\beta y+\frac{\sqrt{y}}{\lambda}R\right)+\left(1-\frac{m+\varepsilon}{2\lambda}\right)\times\left[\frac{1}{\lambda^2}\left(R^2-\lambda\sqrt{y}\,\dot{R}\right)+\left(n+\frac{\nu+1}{2}\right)-\frac{1}{4}\frac{\nu^2-\frac{1}{4}}{y}-\frac{1}{4}y\right], \quad (3.1.26)$$

To make (3.1.26) linear in y, one would suggest the following constraints:

$$V - c\sqrt{y}R = \lambda\gamma y + \lambda\tau, \tag{3.1.27.a}$$

$$\frac{1}{\lambda^2}\left(R^2 - \lambda\sqrt{y}\,\dot{R}\right) = \frac{1}{4}\frac{\nu^2 - \frac{1}{4}}{y} + \xi y + \eta, \tag{3.1.27.b}$$

Where $\{\gamma, \tau, \xi, \eta\}$ are constant parameters.

The next step is that we choose certain values of the parameters $\{\gamma, \tau, \xi, \eta\}$ such that we solve the differential equation for R (Eq. (3.1.27.b)) exactly. As a first example, if we take $\xi = \nu^2 - \frac{1}{4} = \tau = 0, \eta = k^2$, then the general solution of (3.1.27.b) is given below:

$$R(y) = -k\lambda \tanh\left(2k\sqrt{y} - k\lambda\delta\right), \tag{3.1.28}$$

Where δ is a real constant.

Substituting this solution for R back in (3.1.27.b), we write the vector potential as:

$$V(x) = S(x) = \frac{\lambda}{2}\left\{-kc\lambda x \tanh\left[k\lambda(x-\delta)\right] + \gamma\frac{(\lambda x)^2}{2}\right\}, \tag{3.1.29}$$

Thus, W(x) becomes:

$$W(x) = -\frac{\lambda}{2}\left\{2k\tanh\left[k\lambda(x-\delta)\right] + \lambda c x\right\}, \tag{3.1.30}$$

The function G(y) associated with this case for $\nu = \frac{1}{2}$ can be written as:

$$G(y)=\left\{\gamma+\frac{c}{2}-\frac{1}{4}\left(1-\frac{m+\varepsilon}{2\lambda}\right)\right\}y+\frac{m-\varepsilon}{2\lambda}-\frac{c}{2}+\left(1-\frac{m+\varepsilon}{2\lambda}\right)\times\left[k^2+\left(n+\frac{3}{4}\right)\right], \qquad (3.1.31)$$

Thus, the J-matrix becomes:

$$J_{n,m}=2\lambda\left\{\begin{array}{l}\left[\frac{m}{\lambda}-1+\left(1+k^2\right)\left(1-\frac{m+\varepsilon}{2\lambda}\right)+\left(n+\frac{3}{4}\right)\left\{2\gamma+\frac{1}{2}\left(1-\frac{m+\varepsilon}{2\lambda}\right)\right\}+\frac{c}{4}\right]\delta_{n,m}\\ -\left\{\gamma-\frac{1}{4}\left(1-\frac{m+\varepsilon}{2\lambda}\right)\right\}\left[\sqrt{n(n+\frac{1}{2})}\delta_{n,m+1}+\sqrt{(n+1)(n+\frac{3}{2})}\delta_{n,m-1}\right]\end{array}\right\}, \qquad (3.1.32)$$

The plot of the vector potential versus x for different choices of parameters is shown in figure 12.

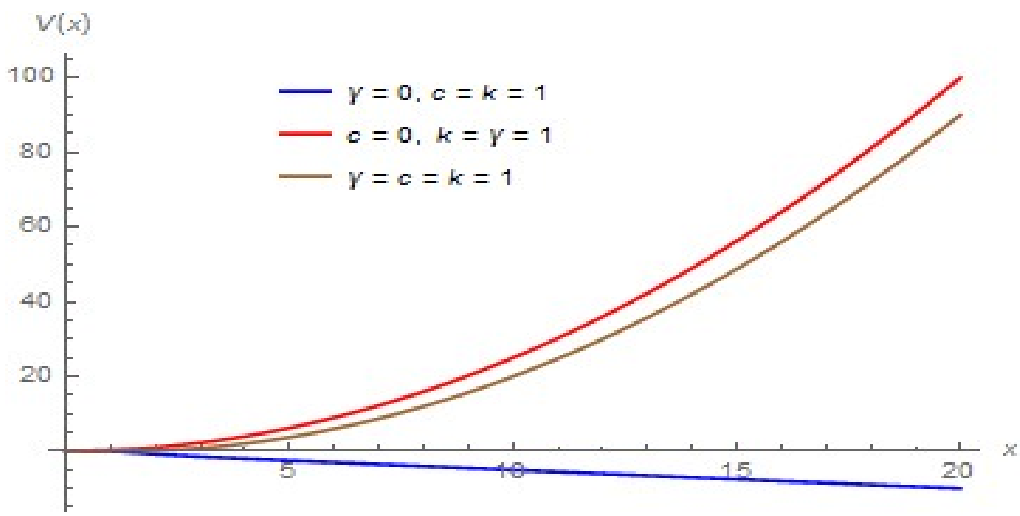

Figure 12: Plot of V(x) for different choices of parameters as indicated in the figure where we took λ=1.

The associated three-term recursion relation reads:

$$\begin{array}{l}\left[\frac{m}{\lambda}-1+\left(1+k^2\right)\left(1-\frac{m+\varepsilon}{2\lambda}\right)+\frac{c}{4}\right]f_n=-2\left(n+\frac{3}{4}\right)\left\{\gamma+\frac{1}{4}\left(1-\frac{m+\varepsilon}{2\lambda}\right)\right\}f_n\\ \qquad+\left\{\gamma-\frac{1}{4}\left(1-\frac{m+\varepsilon}{2\lambda}\right)\right\}\left[\sqrt{n(n+\frac{1}{2})}f_{n-1}+\sqrt{(n+1)(n+\frac{3}{2})}f_{n+1}\right]\end{array}, \qquad (3.1.33)$$

If we consider the simplest case where γ=0, then we write (3.1.33) as follows:

$$zf_n = -\left(n+\frac{7}{4}+2k^2\right)f_n - \frac{1}{2}\left[\sqrt{n(n+\frac{1}{2})}f_{n-1} + \sqrt{(n+1)(n+\frac{3}{2})}f_{n+1}\right], \qquad (3.1.34)$$

Where: $z = 2\left[\frac{m}{\lambda} - 1 + \frac{c}{4}\right] / \left(1 - \frac{m+\varepsilon}{2\lambda}\right)$.

We plot the four lowest energies vs the range of k values as in the figure below:

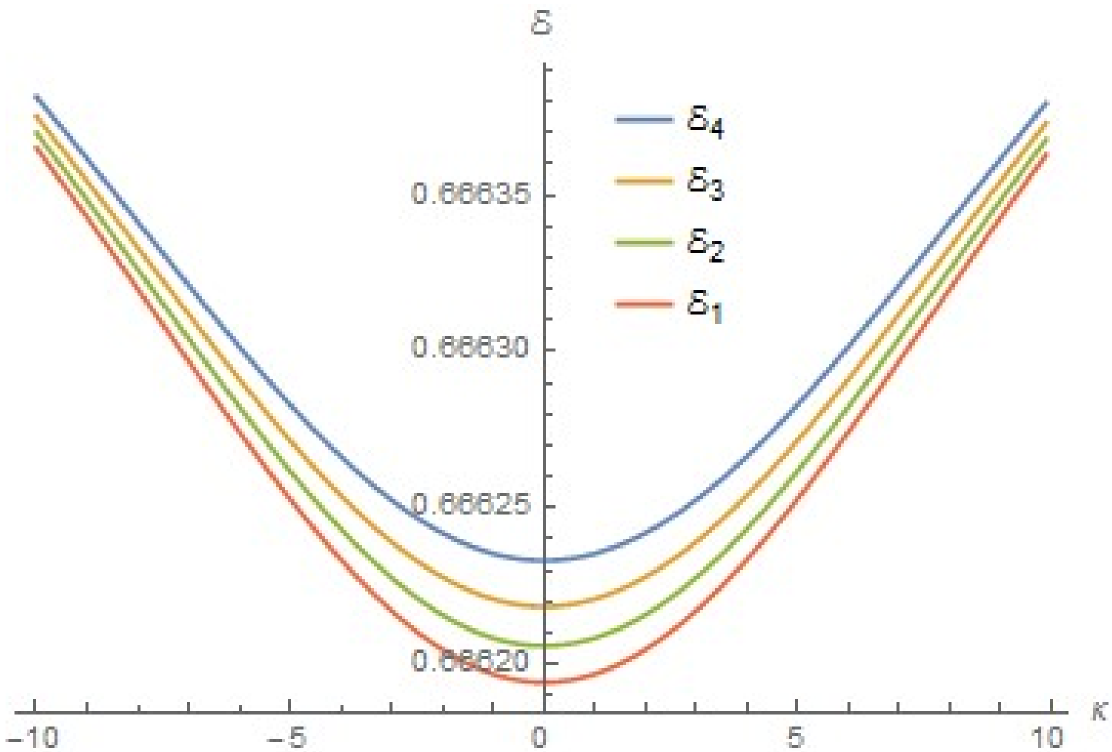

Figure 13: The energy spectrum vs k. Here we took m=c=1, λ=1.5. As we can see, each energy curve goes through a transition at certain value of k which changes the behavior of the curve.

The table below shows few of the polynomials that satisfy (3.1.34):

Table 2: Few polynomials associated with (3.1.34) where we took $f_{-1}(z)=0, f_0(z)=1$ and the variable z is related to the energy by $z=2\left[\frac{m}{\lambda}-1+\frac{c}{4}\right]/\left(1-\frac{m+\varepsilon}{2\lambda}\right)$.

n	$f_n(z)$
1	$-\frac{7}{\sqrt{6}}-4\sqrt{\frac{2}{3}}k^2-2\sqrt{\frac{2}{3}}z$
2	$\frac{71+64k^4+8z(9+2z)+16k^2(9+4z)}{2\sqrt{30}}$
3	$\frac{-925-1284z-8(64k^6+24k^4(11+4z)+2z^2(33+4z)+3k^2(107+8z(11+2z)))}{12\sqrt{35}}$

Also, we plotted the ground state spinor wavefunction as follows:

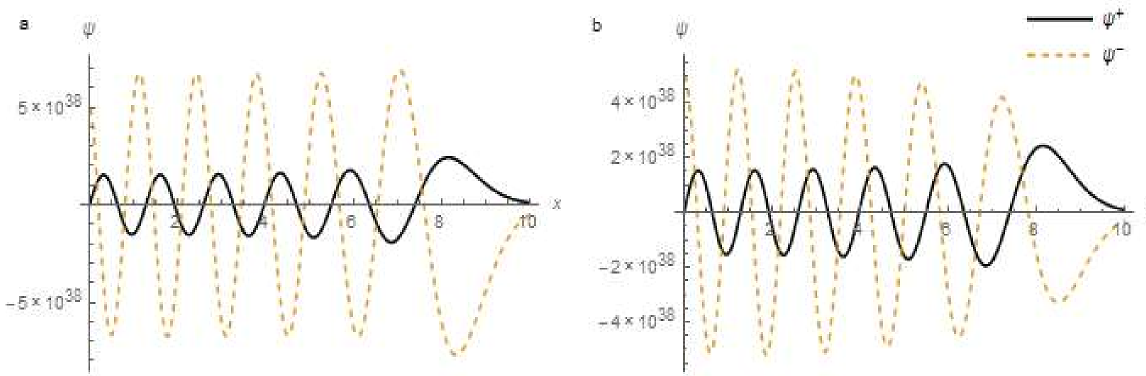

Figure 14: A plot of the spinor wavefunction components for the ground state (2.0) for k=3 (a) and for k=1 (b). Here we took m=c=1 and λ=1.5

However, if one considers the general case of (3.1.33) then we can write the recursion relation associated with this most general case as follows:

$$H|f\rangle=\frac{\varepsilon}{2\lambda}\Omega|f\rangle, \tag{3.1.35}$$

where,

$$H_{n,m}=\left\{\left(1-\tfrac{m}{2\lambda}\right)\left(1+k^2\right)+\left(n+\tfrac{3}{4}\right)\left[2\gamma+\tfrac{1}{2}\left(1-\tfrac{m}{2\lambda}\right)\right]+\frac{m}{\lambda}+\frac{c}{4}-1\right\}\delta_{n,m} \\ -\left[\gamma-\tfrac{1}{4}\left(1-\tfrac{m}{2\lambda}\right)\right]\left[\sqrt{n\left(n+\tfrac{1}{2}\right)}\delta_{n,m+1}+\sqrt{(n+1)\left(n+\tfrac{3}{2}\right)}\delta_{n,m-1}\right], \tag{3.1.36}$$

and,

$$\Omega_{n,m}=\left[\left(1+k^2\right)+\tfrac{1}{2}\left(n+\tfrac{3}{4}\right)\right]\delta_{n,m}+\tfrac{1}{4}\left[\sqrt{n\left(n+\tfrac{1}{2}\right)}\delta_{n,m+1}+\sqrt{(n+1)\left(n+\tfrac{3}{2}\right)}\delta_{n,m-1}\right], \tag{3.1.37}$$

The eigenvalue equation (3.1.35) can be converted to the following simple form:

$$\Omega^{-1}H\,|\,f\rangle=\frac{\varepsilon}{2\lambda}|\,f\rangle, \tag{3.1.38}$$

We plotted the four lowest energy eigenvalues versus the range of values of the interaction potential parameter γ as follows:

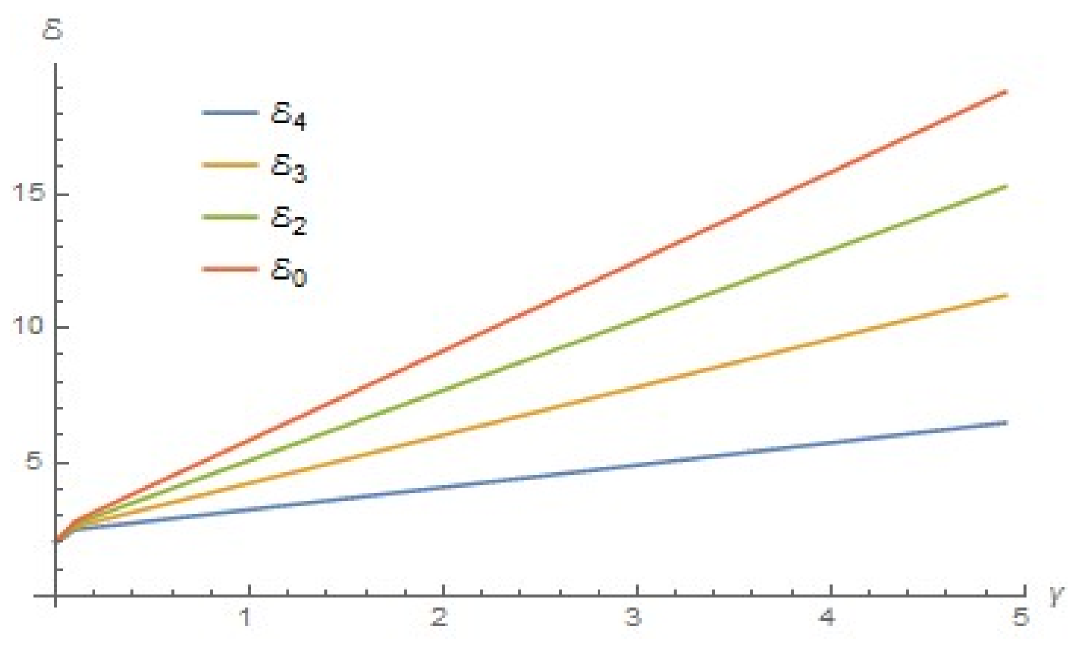

Figure 15: Plot of the four lowest energies vs γ. Here we took m=1, c=4, k=2, and λ=1.5.

It is clear that the general problem is a little harder since the energy eigenvalue appears in all the terms in the recursion relation (generalized eigenvalue problem). Fortunately, we can generate solutions to the recursion relation to any order with high accuracy. As an illustration, we plotted the spinor wavefunction components for the ground state and the 1st excited state for special choices of parameters indicated in the figure below:

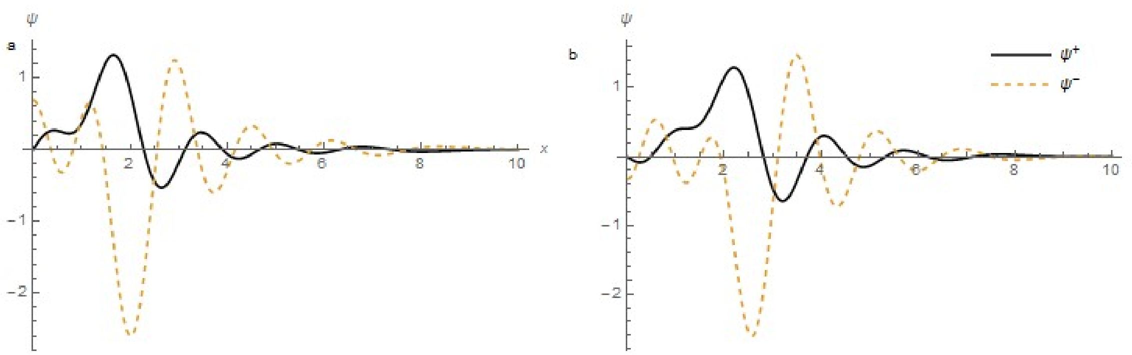

Figure 16: Plot of the spinor wavefunction components. (a) for the ground state (3.23157) and (b) for the first excited state (4.21711). Here we took m=c=k=γ=1, and λ=1.5.

Another case we take here is by choosing the parameters as $\xi = \eta = 0$. Then the general solution of Eq. (3.1.27.b) is given below:

$$R(y) = \frac{\lambda}{4\sqrt{y}}\left(-(1+2\nu) + \frac{4\nu\delta}{y^{\nu}+\delta}\right), \tag{3.1.39}$$

Where δ is non-negative number and ν is an even positive integer to avoid having singularities within the domain of the Hamiltonian.

This is similar to what we have got in the first example in the case where S=V=0. But in this case V and S are not zeroes and they are given below:

$$V(x) = S(x) = \lambda\left\{c\frac{\nu\delta}{y^{\nu}+\delta} + \gamma y\right\}, \tag{3.1.40}$$

$$W(x) = \lambda\left\{\frac{1}{4\sqrt{y}}\left(-(1+2\nu) + \frac{4\nu\delta}{y^{\nu}+\delta}\right) - c\sqrt{y}\right\}, \tag{3.1.41}$$

Where we took $\tau = -c\frac{2\nu+1}{4}$.

We plot the potentials for special choices of parameters as shown in the figure below:

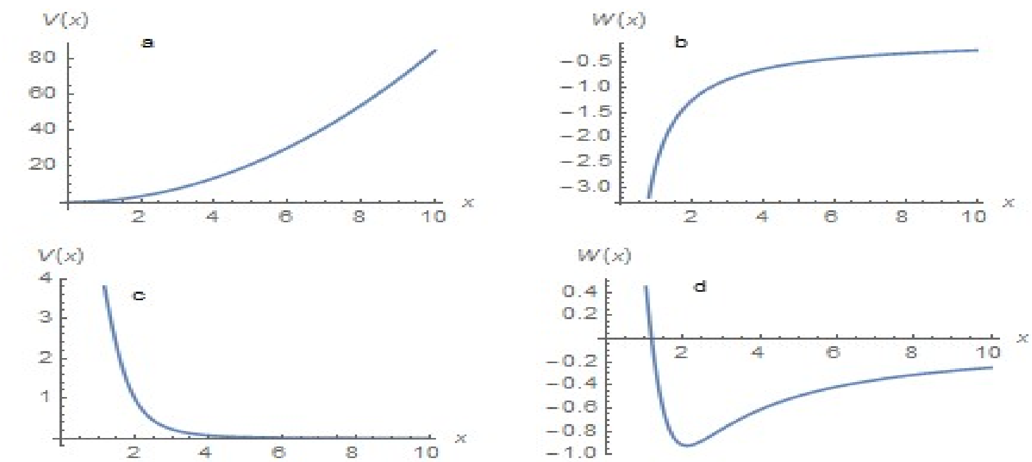

Figure 17: Plot of V(x) and W(x). For (a, b) we took δ=0, γ=1, ν=2. For (b, c) we took γ=0,c=2, δ=1 and λ=1.5.

Now, we can write G(y) for this case as:

$$G(y) = \left(\gamma + \frac{c}{2} - \frac{1}{4}\left(1 - \frac{m+\varepsilon}{2\lambda}\right)\right)y + \frac{m-\varepsilon}{2\lambda} + \left(1 - \frac{m+\varepsilon}{2\lambda}\right)\left(n + \frac{\nu+1}{2}\right), \tag{3.1.42}$$

Thus, the J-matrix becomes:

$$J_{n,m}=2\lambda\left\{\begin{array}{l}\left[\frac{m}{\lambda}-1-4\gamma+\left(n+\frac{\nu+5}{2}\right)\left(2\gamma+\frac{1}{2}\left(1-\frac{m+\varepsilon}{2\lambda}\right)\right)+c\frac{\nu+1}{2}\right]\delta_{n,m}\\-\left(\gamma-\frac{1}{4}\left(1-\frac{m+\varepsilon}{2\lambda}\right)\right)\left[\sqrt{n(n+\nu)}\delta_{n,m+1}+\sqrt{(n+1)(n+\nu+1)}\delta_{n,m-1}\right]\end{array}\right\}, \tag{3.1.43}$$

Now, the three-term recursion relation associated with these potentials is written as follows:

$$\begin{aligned}\left[\frac{m}{\lambda}-1-4\gamma+c\frac{\nu+1}{2}\right]f_n &= -\left(n+\frac{\nu+5}{2}\right)\left(2\gamma+\frac{1}{2}\left(1-\frac{m+\varepsilon}{2\lambda}\right)\right)f_n\\&\quad+\left(\gamma-\frac{1}{4}\left(1-\frac{m+\varepsilon}{2\lambda}\right)\right)\left[\sqrt{n(n+\nu)}f_{n-1}+\sqrt{(n+1)(n+\nu+1)}f_{n+1}\right]\end{aligned}, \tag{3.1.44}$$

For the easiest case where γ=0, we plot the energy spectrum versus c for the four lowest energy states as follows:

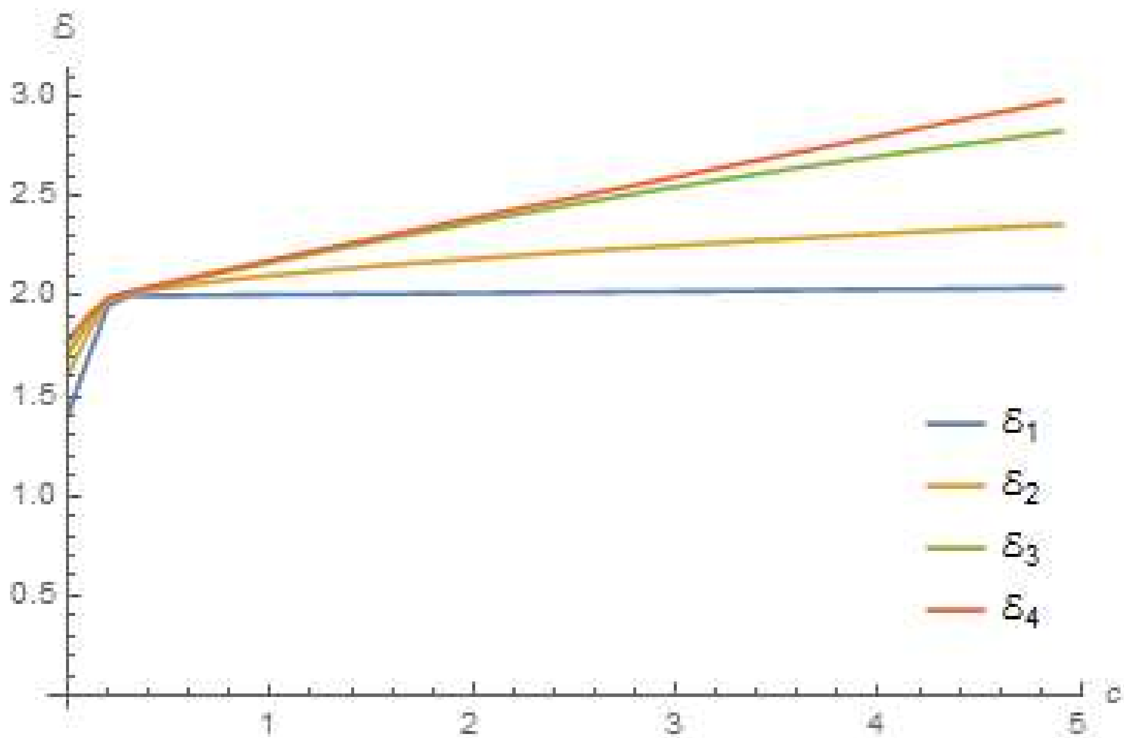

Figure 18: Plot of the four lowest energies vs c. We took m=c=1,=2, λ=1.5 and γ=0.

Another case one would consider here is the case when c=0, which makes S and V to be the harmonic oscillator potentials. We plot the same spectrum as done previously:

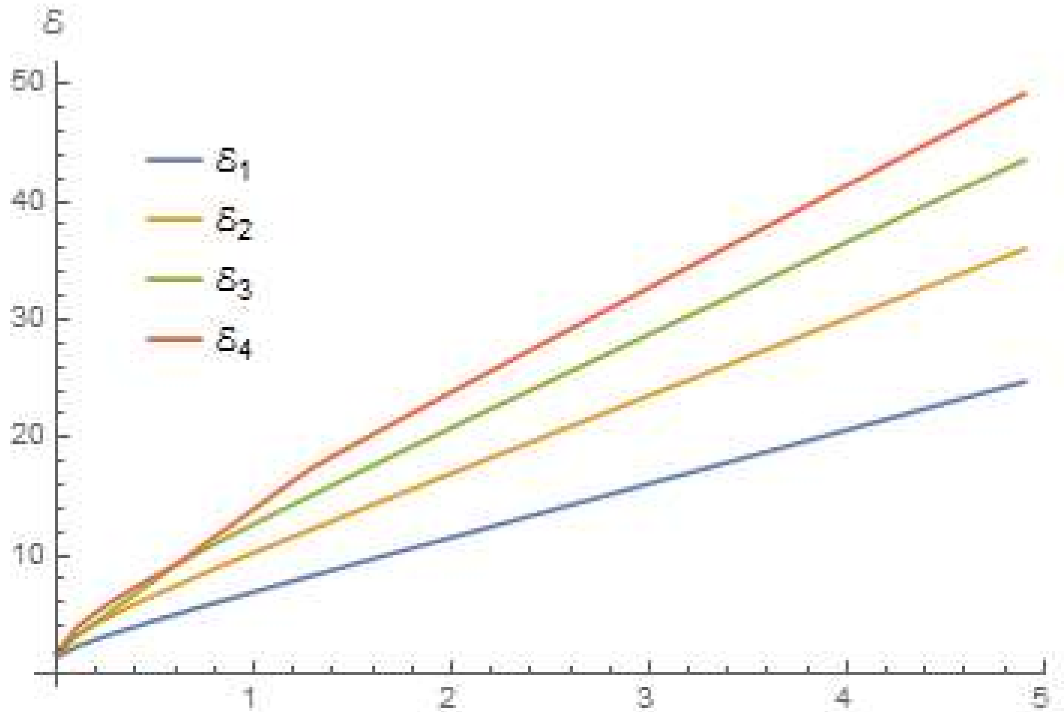

Figure 19: Plot of the four lowest energies vs γ. We took m=c=1, ν=2, =1.5 and c=0.

We plot the wavefunction for the first two lowest states for c=0 as follows:

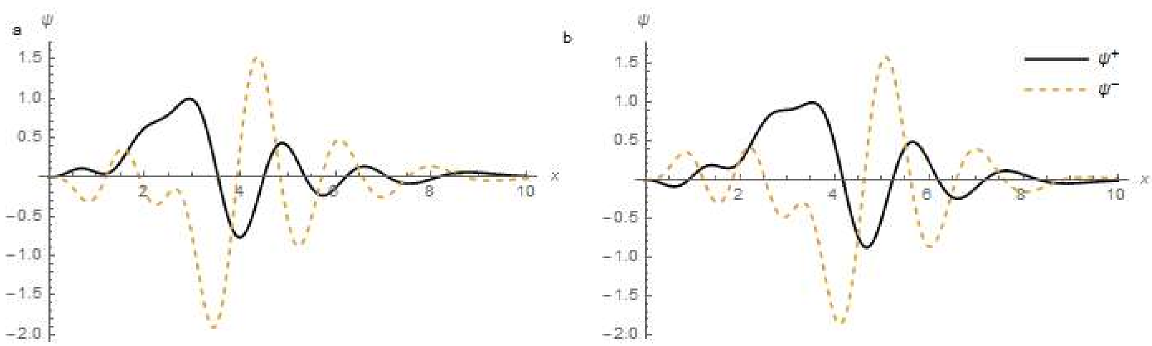

Figure 20: Plot of the spinor wavefunction components. (a) for the ground state (6.92838) and (b) for the first excited state (10.26010). Here we took ν=2m=2, λ=1.5, c=δ=0 and γ=1.

One last case we mention here for the spin symmetric coupling V=S and nonzero W is obtained by taking $\eta = 0, \xi = \frac{1}{4}, \nu = 1$ which gives another exact solution for R given below:

$$R(y) = \frac{\lambda}{4\sqrt{y}}\left(1 - 2y + \delta\frac{4y}{e^y + \delta}\right), \tag{3.1.45}$$

where δ is a positive real number.

Using (3.1.27), we can write the vector and scalar potentials as:

$$V(x) = S(x) = \frac{\lambda}{8}(\lambda x)^2\left\{\frac{2c\delta}{\exp\left((\lambda x)^2/4\right) + \delta} + (2\gamma - c)\right\}, \tag{3.1.46}$$

where we took $\tau = -\frac{c}{4}$. The pseudopotential is calculated using Eq. (2.11), which gives:

$$W(x) = \frac{1}{2x}\left(1 - (2c+1)\frac{(\lambda x)^2}{2} + \delta\frac{(\lambda x)^2}{\exp\left((\lambda x)^2/4\right) + \delta}\right), \tag{3.1.47}$$

Now, we can write the function G(y) for this case as follows:

$$G(y) = \left(\gamma + \frac{c}{2}\right)y + \left(1 - \frac{m+\varepsilon}{2\lambda}\right)(n+2) + \frac{m}{\lambda} - 1 - c, \tag{3.1.48}$$

Thus, the J-matrix for this special case reads as:

$$J_{n,m} = 2\lambda\left\{\begin{array}{l}\left[\frac{m-\varepsilon}{2\lambda} + (n+1)\left(2\gamma + \left(1 - \frac{m+\varepsilon}{2\lambda}\right)\right)\right]\delta_{n,m} \\ -\gamma\left[\sqrt{n(n+1)}\delta_{n,m+1} + \sqrt{(n+1)(n+2)}\delta_{n,m-1}\right]\end{array}\right\}, \tag{3.1.49}$$

The three-term recursion relation of the expansion coefficients becomes:

$$\left[\frac{m-\varepsilon}{2\lambda}\right]f_n = -(n+1)\left(2\gamma+\left(1-\frac{m+\varepsilon}{2\lambda}\right)\right)f_n \\ +\gamma\left[\sqrt{n(n+1)}\,f_{n-1}+\sqrt{(n+1)(n+2)}\,f_{n+1}\right], \tag{3.1.50}$$

We plot the four lowest energy eigenvalues versus γ as shown below:

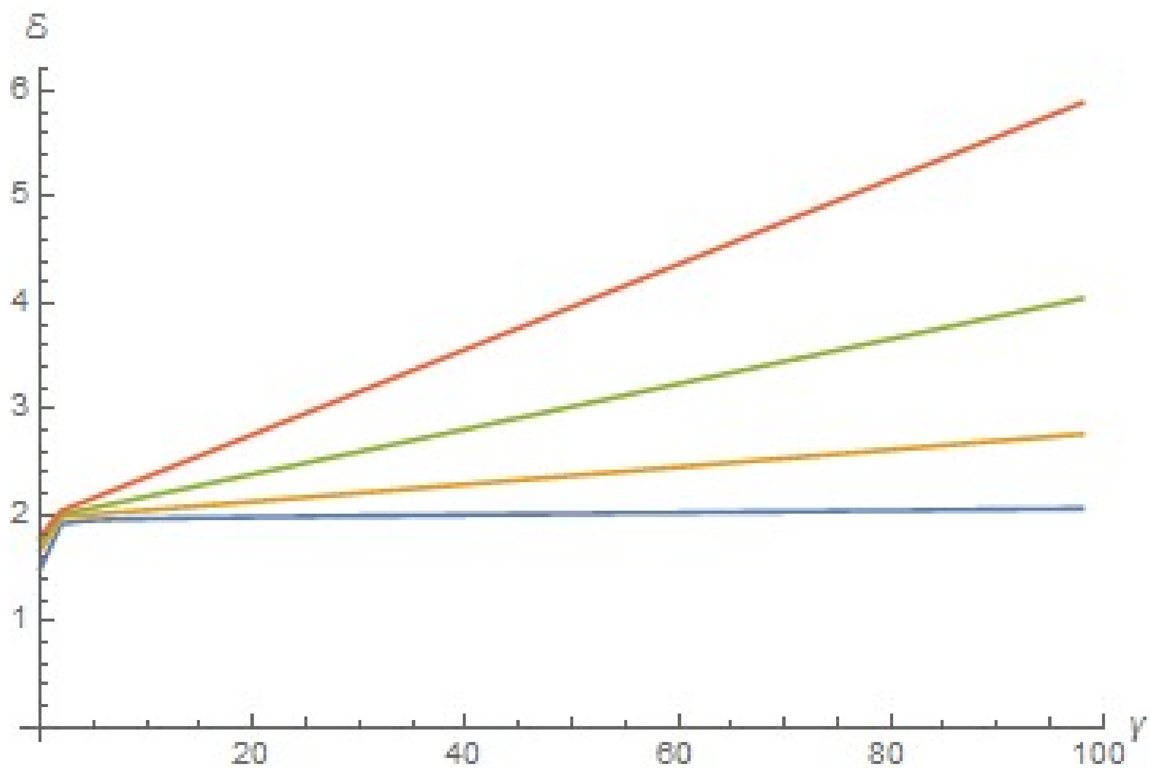

Figure 21: Plot of the four lowest energy eigenvalues versus γ. Here we took m=ν=1 and λ=1.5.

The next page shows a plot of the spinor wavefunction components for the two lowest energy states:

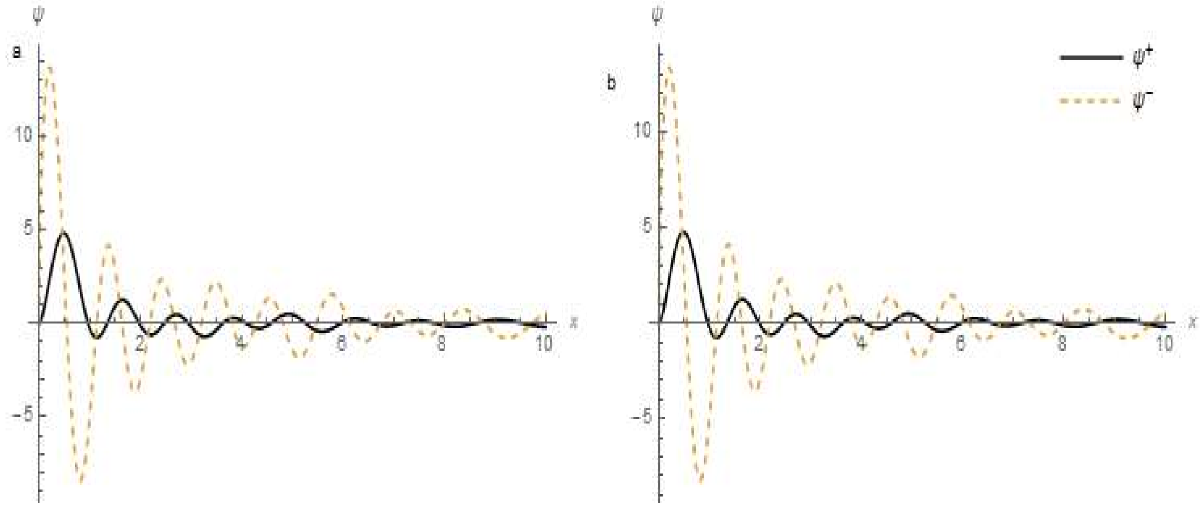

Figure 22: Plot of the spinor wavefunction components. (a) for the ground state (1.88396) and (b) for the 3[rd] excited state (1.99411). Here we took m=γ=δ=1 and λ=1.5.

In conclusion, we have considered different couplings and for each case we plotted the energy spectrum and the spinor wavefuncion. The recursion relations have been solved exactly for two cases in which we have S=V=0 , where the expansion coefficients were written in terms of Meixner-Pollaczek orthogonal polynomials. The other recursion relations for the other cases have been calculated but no closed forms have been found. The cases studied in this section are not the only solvable potentials within this basis, other potentials can still be considered by studying different coupling conditions. In the next section we will consider the class of solvable potentials in the space in which the Hamiltonian is defined on a finite interval (Jacobi basis). We will almost follows the same procedure but the difference will be in the basis functions and hence the J-matrix.

3.2 Solvable potentials within Jacobi Basis

This section will introduce the new solvable potentials within Jacobi basis, that is, the space in which the Hamiltonian is defined on a finite interval of the real line. We will follow here almost the same treatment done for the Laguerre basis. So, first of all, we consider the case where S=V=0, W is nonzero. This makes G(y) takes the following form:

$$G(y)=\frac{m-\varepsilon}{2\lambda}-c\left[\beta-\alpha-(\alpha+\beta)y+\frac{\sqrt{1-y^2}}{\lambda}R\right]+\left(1-\tfrac{m+\varepsilon}{2\lambda}\right)\left\{\left(n+\tfrac{\mu+\nu+1}{2}\right)^2\right. ,$$
$$\left.-\frac{1}{4}\left(\mu^2+\nu^2-\frac{1}{2}\right)-\frac{1}{4}\left[\mu^2-\frac{1}{4}\right]\frac{1+y}{1-y}-\frac{1}{4}\left[\nu^2-\frac{1}{4}\right]\frac{1-y}{1+y}+\frac{1}{\lambda^2}\left(R^2-y'\dot{R}\right)\right\} \qquad (3.2.1)$$

where we took the coordinate transformation $y(x)=\sin(\lambda x)$.

To make (3.2.1) linear in y, we consider the simple case where R=0 and the parameter choice $\mu^2=\nu^2=\frac{1}{4}$. This makes G(y) take the form:

$$G(y)=c\frac{\mu+\nu+1}{2}y+\frac{m-\varepsilon}{2\lambda}+c\frac{\mu-\nu}{2}+\left(1-\tfrac{m+\varepsilon}{2\lambda}\right)\left(n+\tfrac{\mu+\nu+1}{2}\right)^2, \qquad (3.2.2)$$

The J-Matrix now reads as follows:

$$J_{n,m}=2\lambda\left\{\left[\frac{m-\varepsilon}{2\lambda}+\left(1-\tfrac{m+\varepsilon}{2\lambda}\right)\left(n+\tfrac{\mu+\nu+1}{2}\right)^2\right]\delta_{n,m}-\tfrac{c}{4}\left[\delta_{n,m-1}+\delta_{n,m+1}\right]\right\}, \qquad (3.2.3)$$

Now, Eq. (2.11) suggests the following solution for the pseudopotential:

$$W(x)=-c\lambda\cos(\lambda x), \qquad (3.2.4)$$

The three-term recursion relation reads:

$$\frac{\varepsilon}{2\lambda} f_n = \left[\frac{m-\lambda}{\lambda a_n} + \left(1 - \frac{m}{2\lambda}\right) \right] f_n - \frac{c}{4a_n} \left[f_{n-1} + f_{n+1} \right], \tag{3.2.5}$$

where $a_n = \left\{ \left(n + \frac{\mu+\nu+1}{2} \right)^2 + 1 \right\}$.

For the moment, we do not know if these recursion relations can be compared to a well-known class of orthogonal polynomials or may lead to a new class of polynomials. The next figure shows the behavior of the four lowest energy eigenvalues versus the coupling parameter c for different parameters. Also, we tabulated the first few polynomials for these recursion relations in the next table.

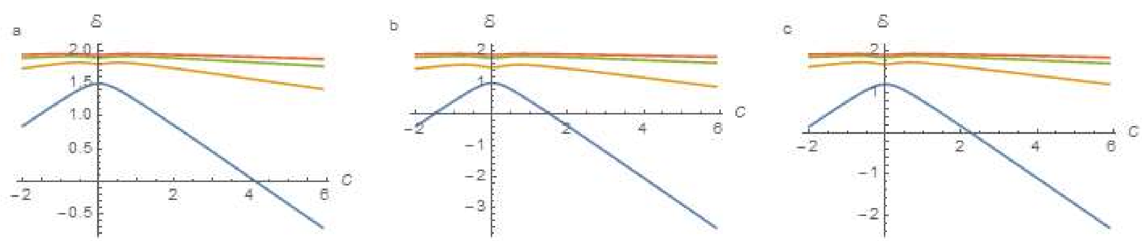

Figure 23: Plot of the four lowest energy states versus c. (a) for the case where $\mu = \nu = -\frac{1}{2}$, (b) for $\mu = \nu = \frac{1}{2}$ and (c) for $\mu = -\nu = \pm\frac{1}{2}$. Here we took m=1, λ=1.5.

Table 3: First few polynomial solutions of the recursion relation (3.2.5). Here we took c=m=1, λ=1.5.

n	$f_n(z)$
1	$4 - 8z$
2	$47 - 176z + 160z^2$
3	$1186.67 - 6330.67z + 11093.34z^2 - 6400z^3$

The spinor wavefunction components for different choices of parameters are plotted below:

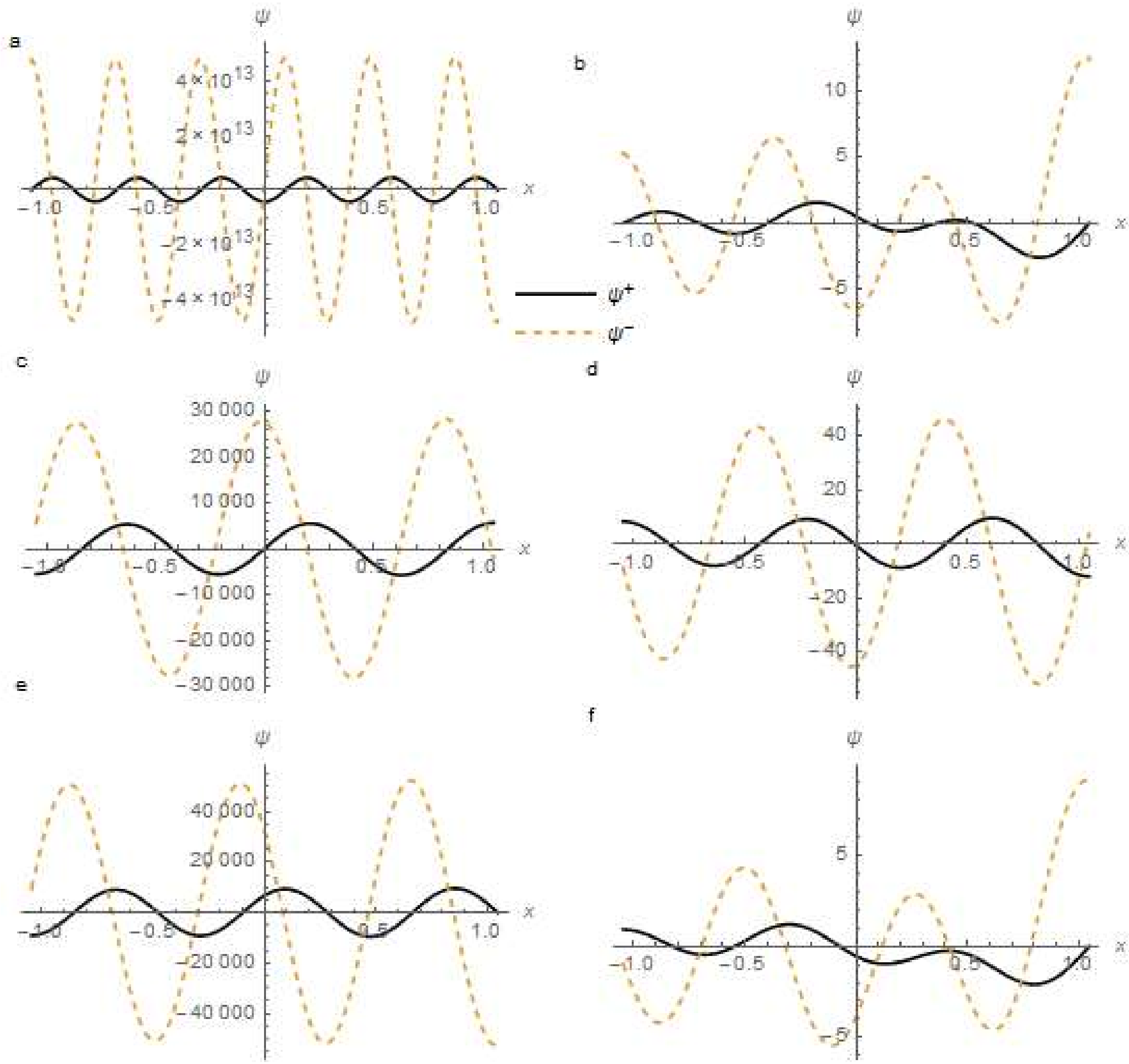

Figure 24: A plot of the un-normalized spinor wavefunction components for different choices of parameters. (a) and (b) for the ground state (1.23503) and the first excited state (1.80503) for the case when $\mu = \nu = \frac{1}{2}$, respectively. (c) and (d) are for the ground state (0.42259) and the first excited state (1.57301) for the case $\mu = \nu = -\frac{1}{2}$, respectively. (e) and (f) for the ground state (0.78178) and the first excited state (1.72359) for the case $\mu = -\nu = \frac{1}{2}$, respectively. In all these cases we took $m = c = 1$ and $\lambda = 1.5$.

Another way to make (3.2.1) linear in y is by considering the particular solution of R suggested by requiring the linearity of $\frac{\sqrt{1-y^2}}{\lambda}R$, that is:

$$R(y)=\frac{\lambda(\gamma y+\tau)}{\sqrt{1-y^2}}, \tag{3.2.6}$$

Where $\{\gamma,\tau\}$ are dimensionless constant parameters.

Considering $\mu=\nu$ and using the solution of R given above in Eq. (3.2.1), we restrict the parameters in the problem to achieve the linearity of G(y) as follows:

$$\gamma=\frac{1}{2},\tau^2=\mu^2=\nu^2=\frac{3}{4}, \tag{3.2.7}$$

Using Eq. (2.11) we can write the associated pseudopotential as follows:

$$W^{\pm}(x)=\frac{\lambda}{2}\left[\tan(\lambda x)\pm\sqrt{3}\sec(\lambda x)-2c\cos(\lambda x)\right], \tag{3.2.8}$$

Now, we can write G(y) as follows:

$$G(y)=c\left(\frac{\mu+\nu+1}{2}-\gamma\right)y+\frac{m-\varepsilon}{2\lambda}-c\left[\frac{\nu-\mu}{2}+\tau\right]+\left(1-\tfrac{m+\varepsilon}{2\lambda}\right)\left[\left(n+\tfrac{\mu+\nu+1}{2}\right)^2-\frac{1}{4}\right], \tag{3.2.9}$$

The associated J-matrix will read as follows:

$$J_{n,m}=2\lambda\left\{\begin{array}{l}\left[\frac{m-\varepsilon}{2\lambda}-c\left[\frac{\nu-\mu}{2}+\tau\right]+\left(1-\frac{m+\varepsilon}{2\lambda}\right)\left[\left(n+\frac{\mu+\nu+1}{2}\right)^2-\frac{1}{4}\right]+\frac{2nc(\nu-\mu)(n+\mu+\nu+1)}{(2n+\mu+\nu)(2n+\mu+\nu+2)}\right]\delta_{n,m}\\ -c(2\gamma+1)\left[\begin{array}{l}\frac{1}{(2n+\mu+\nu+2)}\sqrt{\frac{(n+1)(n+\mu+\nu+1)(n+\mu+1)(n+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+3)}}\delta_{n,m-1}\\ +\frac{1}{(2n+\mu+\nu)}\sqrt{\frac{n(n+\mu+\nu)(n+\nu)(n+\mu)}{(2n+\mu+\nu-1)(2n+\mu+\nu+1)}}\delta_{n,m+1}\end{array}\right]\end{array}\right\}, \tag{3.2.10}$$

The plot of the pseudopotential is given in the following figure:

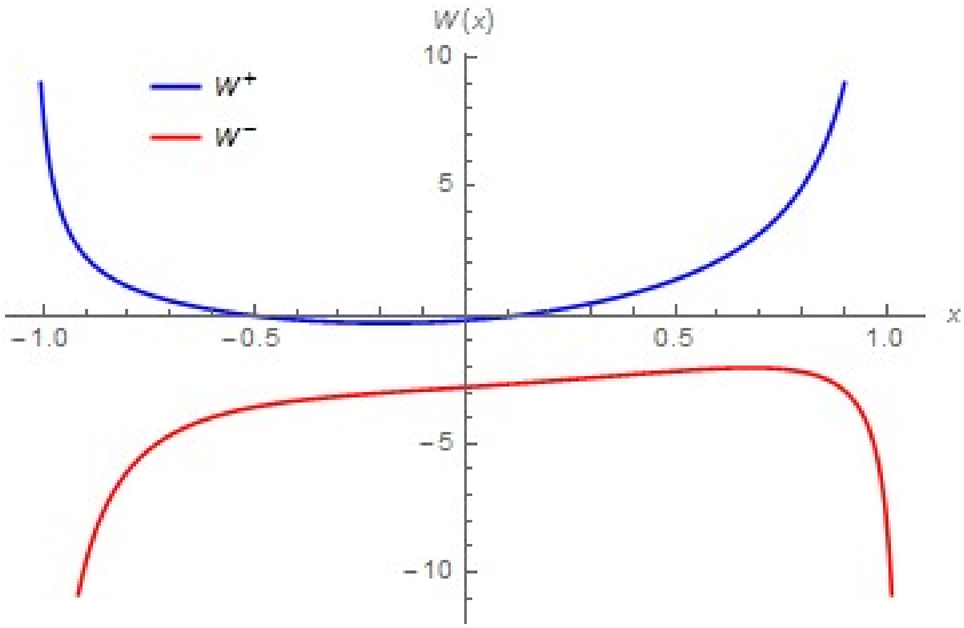

Figure 25: Plot of the pseudopotential for c=1, λ=1.5.

The three-term recursion relation associated with (3.2.10) is written as follows:

$$\begin{aligned}\left[\frac{m-\varepsilon}{2\lambda}-c\left[\frac{\nu-\mu}{2}+\tau\right]\right]f_n=&-\left\{\left(1-\tfrac{m+\varepsilon}{2\lambda}\right)\left[\left(n+\tfrac{\mu+\nu+1}{2}\right)^2-\frac{1}{4}\right]+\frac{2nc(\nu-\mu)(n+\mu+\nu+1)}{(2n+\mu+\nu)(2n+\mu+\nu+2)}\right\}f_n\\&+c(2\gamma+1)\left[\begin{aligned}&\frac{1}{(2n+\mu+\nu+2)}\sqrt{\frac{(n+1)(n+\mu+\nu+1)(n+\mu+1)(n+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+3)}}f_{n+1}\\&+\tfrac{1}{(2n+\mu+\nu)}\sqrt{\frac{n(n+\mu+\nu)(n+\nu)(n+\mu)}{(2n+\mu+\nu-1)(2n+\mu+\nu+1)}}f_{n-1}\end{aligned}\right]\end{aligned}\quad, \qquad (3.2.11)$$

Each choice of parameters will have its own recursion relation, energy spectrum and of course its own wavefunction. The next plot shows the behavior of the four lowest energy eigenvalues versus the interaction parameter c for each choice of parameters. The second figure describes the oscillatory behavior of the spinor components for each of these cases in their finite space.

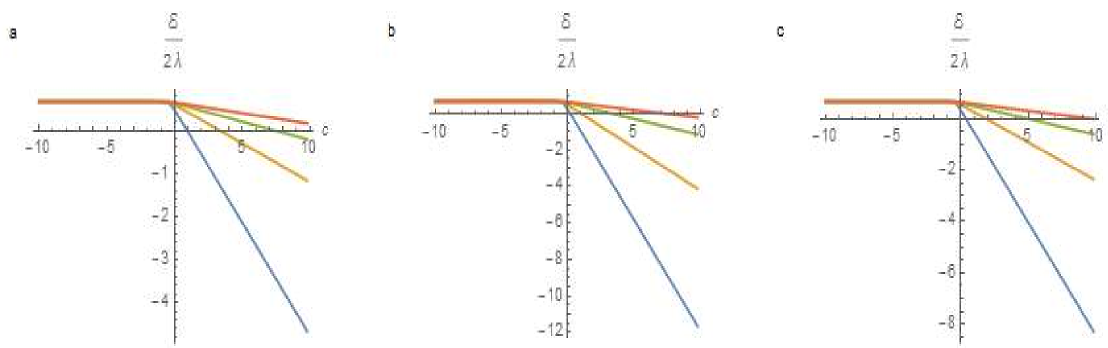

Figure 26: Plot of the four lowest energy states versus c. (a) for the case where $\mu = \nu = \frac{\sqrt{3}}{2}$, (b) for $\mu = \nu = -\frac{\sqrt{3}}{2}$ and (c) for $\mu = -\nu = \pm\frac{\sqrt{3}}{2}$. Here we took m=1, λ=1.5 and $\tau = \frac{\sqrt{3}}{2}$.

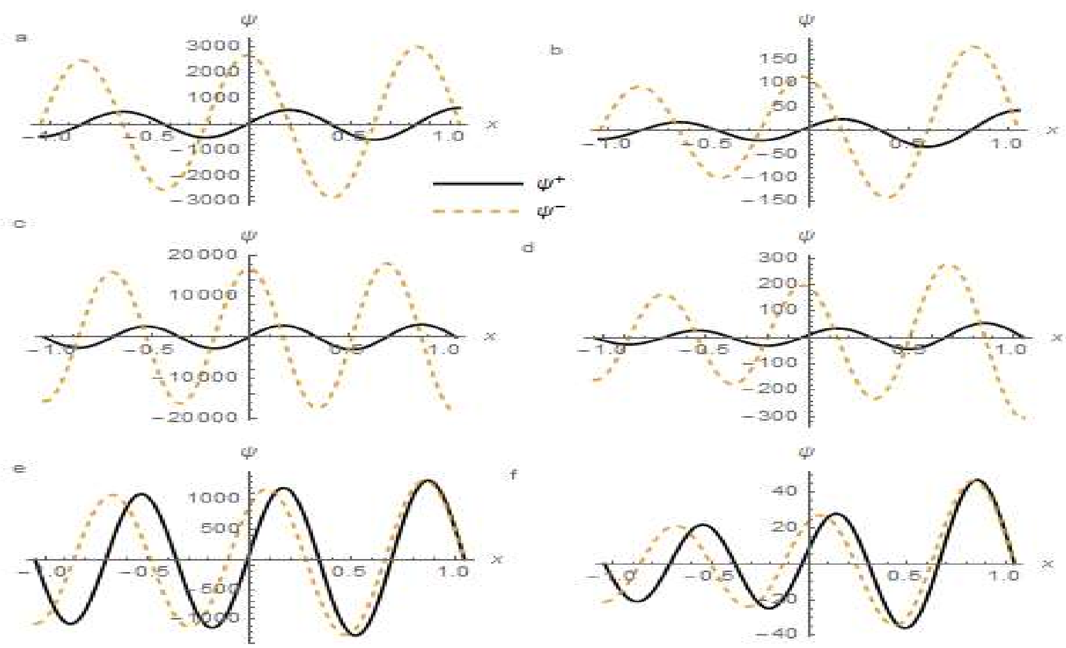

Figure 27: A plot of the spinor wavefunction components for the ground states and the first excited state for different choice of parameters are shown above. (a, b) for the case where $\mu = \nu = \frac{\sqrt{3}}{2}$ the left for the ground state (a) and the right for the 1st excited state (b). (c, d) represents the spinor components for $\mu = \nu = -\frac{\sqrt{3}}{2}$ where (c) for the ground state and (d) for the 1st excited state. Finally, (e, f) are for the case $\mu = -\nu = \pm\frac{\sqrt{3}}{2}$ where (e) for the ground state and (f) for the 1st excited state.

Now, we consider another coupling potential in which W=0, V and S are nonzero. Taking $\mu^2 = \nu^2 = \frac{1}{4}$ and considering the trivial solution of R, that is, R=0, together with $V(x) = V_0 y = V_0 \sin(\lambda x + \delta)$ and $S(x) = k_y = const$, we write G(y) as follows:

$$G(y) = \frac{V_0}{2\lambda}\left[\left(n+\tfrac{\mu+\nu+1}{2}\right)^2 + \frac{\mu+\nu+3}{2}\right] y + \left(1-\tfrac{m'+\varepsilon}{2\lambda}\right)\left(1+\left(n+\tfrac{\mu+\nu+1}{2}\right)^2\right) + \frac{m'}{\lambda} - 1 + \frac{V_0}{2\lambda}\left(\frac{\mu-\nu}{2}\right), \quad (3.2.12)$$

where: $m' = m + k_y$ and δ is a real phase shift.

We are interested in applying this case to graphene (See Appendix D). So, in the next calculations we will consider the massless Dirac equation (m=0). Thus, we can write the J-matrix associated for this special case as follows:

$$J_{n,m} = 2\lambda\left\{\left[\left(1-\tfrac{k_y+\varepsilon}{2\lambda}\right)\left(1+\left(n+\tfrac{\mu+\nu+1}{2}\right)^2\right) + \frac{k_y}{\lambda} - 1\right]\delta_{n,m} + \frac{V_0}{8\lambda}\left[2\left(n+\tfrac{\mu+\nu+1}{2}\right)^2 + 1\right]\left[\delta_{n,m-1} + \delta_{n,m+1}\right]\right\}, \qquad (3.2.13)$$

Where here we have two choices of parameters which are $\mu = \nu = \pm\frac{1}{2}$ or $\mu = -\nu = \pm\frac{1}{2}$.

The three-term recursion relation for this case is written as:

$$\tfrac{\varepsilon}{2\lambda} f_n = \left[\left(1-\tfrac{k_y}{2\lambda}\right) + \frac{k_y - \lambda}{\lambda a_n}\right] f_n + \frac{V_0}{8\lambda}\left[\frac{2a_n - 1}{a_n}\right]\left[f_{n-1} + f_{n+1}\right], \qquad (3.2.14)$$

where,

$$a_n = \left(\left(n+\tfrac{\mu+\nu+1}{2}\right)^2 + 1\right).$$

We plot the lowest four energy eigenvalues versus k_y as shown in the next figure:

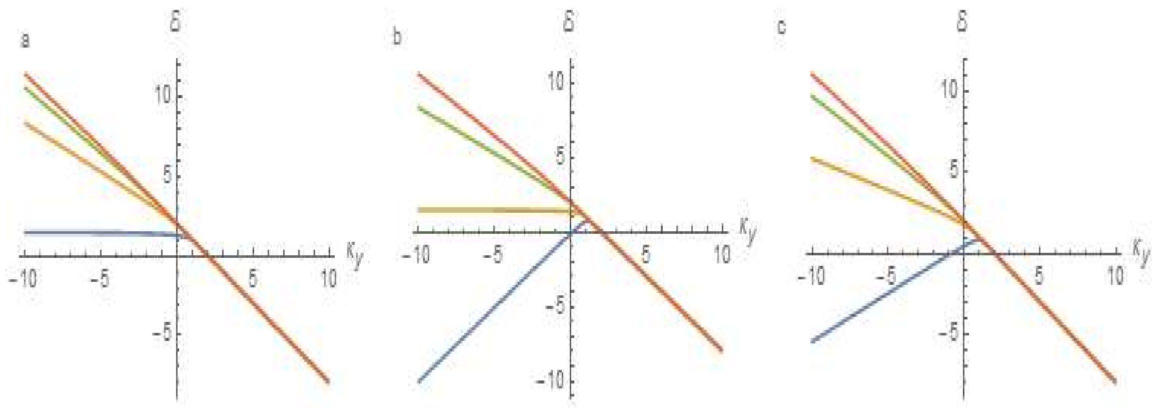

Figure 28: Plot of the four lowest energy eigenvalues versus k_y for different choices of parameters. Starting from the left, (a) for the case where $\mu = \nu = \frac{1}{2}$, (b) for the case $\mu = \nu = -\frac{1}{2}$ and finally (c) for the last case when $\mu = -\nu = \pm\frac{1}{2}$.

Table 4: The first four polynomial solutions of (3.2.14). Here $z = \varepsilon / 2\lambda$, we took m=$k_y$=$V_0$=1 and λ=1.5. The parameter choice is $\mu = \nu = \frac{1}{2}$.

n	$f_n(z)$
1	$-4 + 8z$
2	$15 - 58.67z + 53.34z^2$
3	$-56 + 321.4z - 583.9z^2 + 336.8z^3$
4	$209 - 1573.13z + 4269z^2 - 4956.7z^3 + 2082.3z^4$

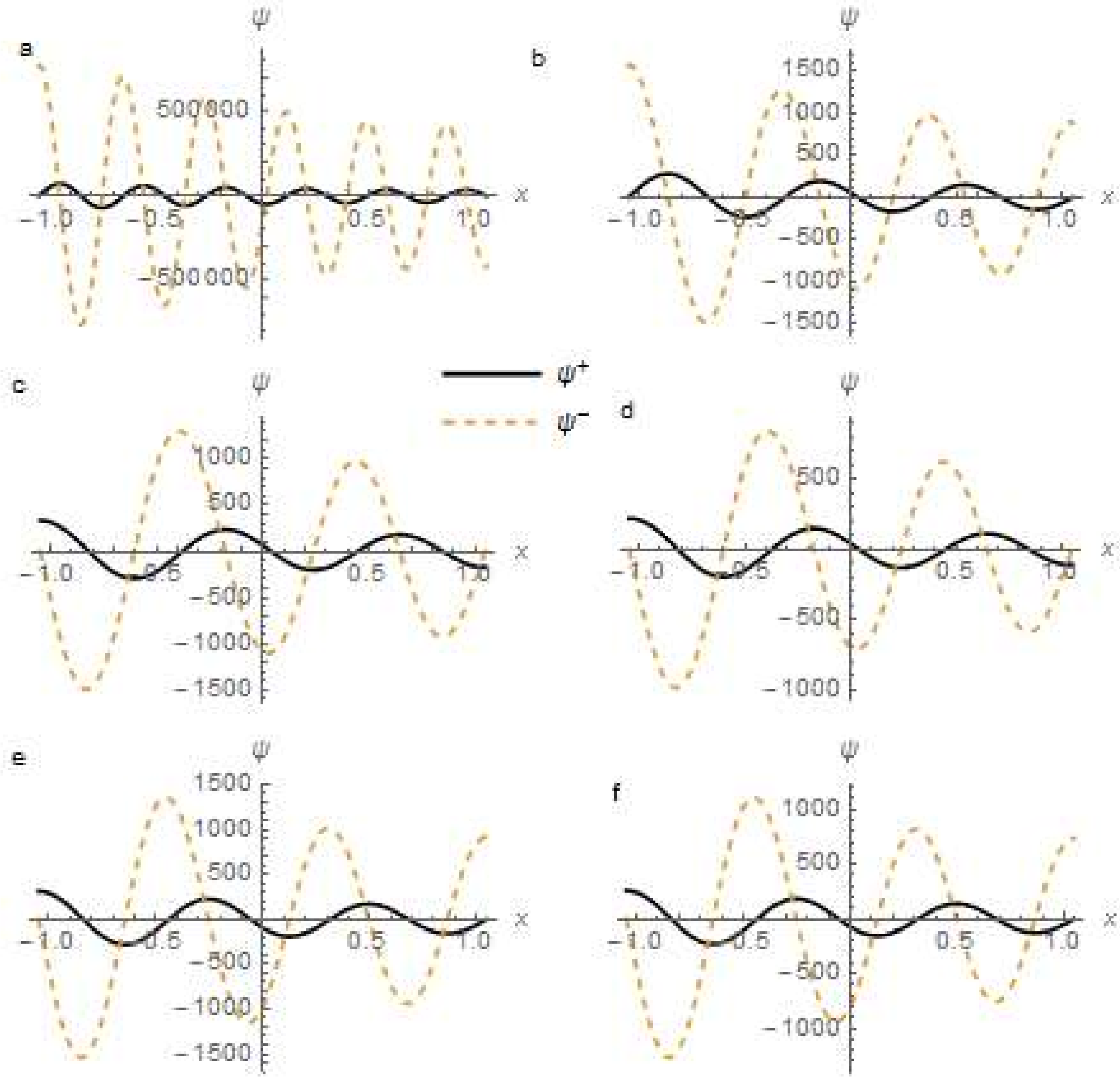

Figure 29: A plot of the un-normalized spinor wavefunction components for different choices of parameters. (a) and (b) for the ground state (0.32379) and the first excited state (0.33340) for the case when $\mu = \nu = \frac{1}{2}$, respectively. (c) and (d) are for the ground state (0.24375) and the first excited state (0.33290) for the case $\mu = \nu = -\frac{1}{2}$, respectively. (e) and (f) for the ground state (0.28625) and the first excited state (0.33326) for the case $\mu = -\nu = \frac{1}{2}$, respectively. In all these cases we took $k_y = V_0 = 1$.

Another example we consider here is the case in which we have spin-symmetric coupling, that is S=V, with W=0. Choosing the trivial solution for R, that is R=0, with $\mu^2 = \nu^2 = \frac{1}{4}$, together with c=0, we write G(y) as follows:

$$G(y) = \frac{V}{\lambda} + \left(1 - \tfrac{m+\varepsilon}{2\lambda}\right)\left\{\left(n + \frac{\mu+\nu+1}{2}\right)^2 + 1\right\} + \frac{m}{\lambda} - 1, \tag{3.2.15}$$

Thus, our choice of the potential must be $V(x) = S(x) = \lambda\gamma y = \lambda\gamma\sin(\lambda x)$, where γ is a dimensionless constant (interaction parameter). Now, we can write the J-matrix for this case as follows:

$$J_{n,m} = 2\lambda\left\{\left[\left(1 - \tfrac{m+\varepsilon}{2\lambda}\right)\left\{\left(n + \tfrac{\mu+\nu+1}{2}\right)^2 + 1\right\} + \frac{m}{\lambda} - 1\right]\delta_{n,m} + \tfrac{\gamma}{2}\left[\delta_{n,m-1} + \delta_{n,m+1}\right]\right\}, \tag{3.2.16}$$

Consequently, the three-term recursion relation is written below:

$$\tfrac{\varepsilon}{2\lambda} f_n = \frac{\left(1 - \tfrac{m}{2\lambda}\right)a_n + \dfrac{m}{\lambda} - 1}{a_n} f_n + \tfrac{\gamma}{2a_n}\left[f_{n-1} + f_{n+1}\right], \tag{3.2.17}$$

Where: $a_n = \left(n + \tfrac{\mu+\nu+1}{2}\right)^2 + 1$.

The recursion relation (3.2.17) is under study. For the moment, we can calculate the spectrum with high accuracy by converting this recursion relation to a matrix form $T\,|\,f\rangle = \frac{\varepsilon}{2\lambda}|\,f\rangle$, where T is a tridiagonal symmetric matrix given below:

$$T_{n,m} = A_n\delta_{n,m} + B_n\left[\delta_{n,m-1} + \delta_{n,m+1}\right], \tag{3.2.18}$$

where: $A_n = \dfrac{\left(1-\frac{m}{2\lambda}\right)a_n + \dfrac{m}{\lambda} - 1}{a_n}$ and $B_n = \frac{\gamma}{2a_n}$.

The eigenvalue $\dfrac{\varepsilon}{2\lambda}$ for the four lowest energy states is plotted versus γ as shown below:

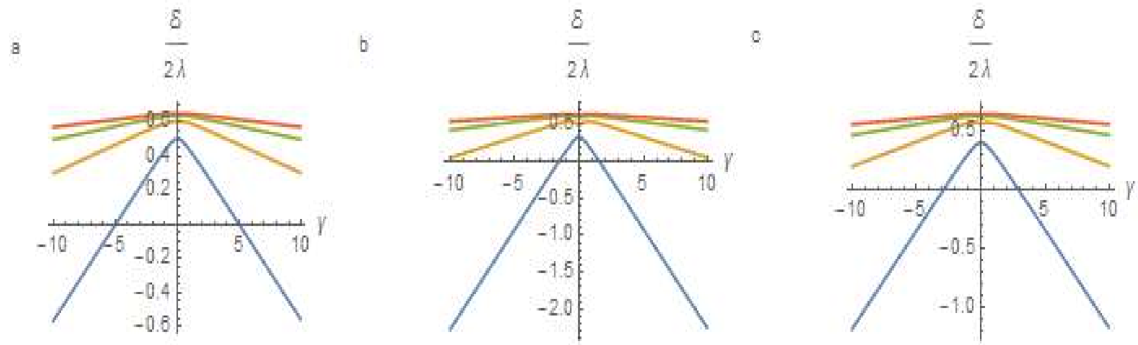

Figure 30: Plot of the four lowest energy states versus the interaction parameter γ. (a) for the case where, $\mu = \nu = \dfrac{1}{2}$, (b) for the case where $\mu = \nu = -\dfrac{1}{2}$ and (c) for the case where $\mu = -\nu = \pm\dfrac{1}{2}$. Here we took m=1, λ=1.5.

The following table shows the first few polynomials of the recursion relation:

Table 5: The first few polynomials of the recursion relation (3.2.17) for the case $\mu = \nu = \dfrac{1}{2}$. Here we took γ=m=1, λ=1.5 and z=ε/2λ.

N	$f_n(z)$
1	$-2 + 4z$
2	$11 - 44z + 40z^2$
3	$-137.34 + 773.34z - 1386.7z^2 + 800z^3$
4	$3010.34 - 21638.7z + 56760z^2 - 64746.67z^3 + 27200z^4$

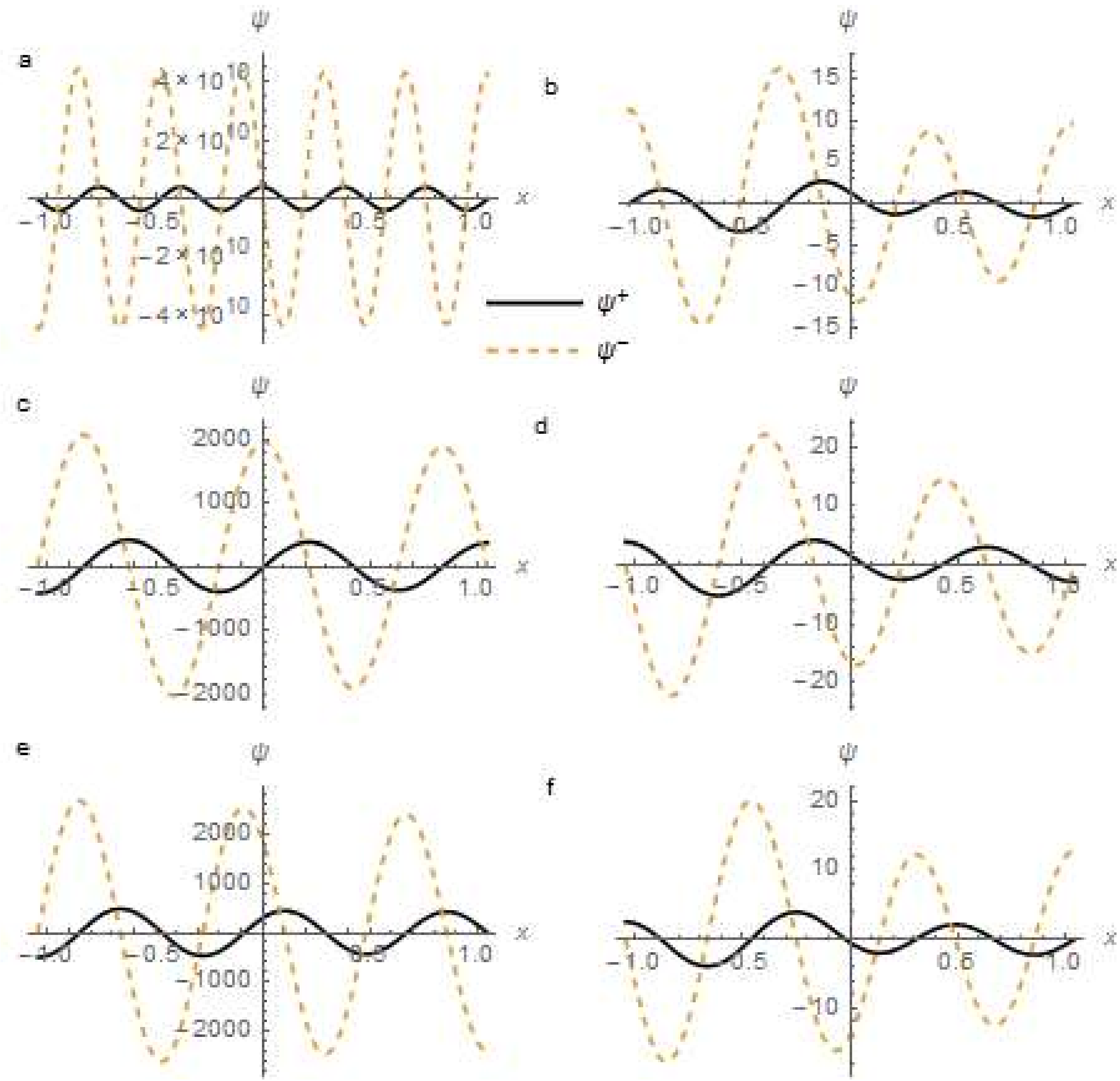

Figure 31: A plot of the un-normalized spinor wavefunction components for different choices of parameters. (a) and (b) for the ground state (1.30349) and the first excited state (1.77959) for the case when $\mu = \nu = \frac{1}{2}$, respectively. (c) and (d) are for the ground state (0.43597) and the first excited state (1.60674) for the case $\mu = \nu = -\frac{1}{2}$, respectively. (e) and (f) for the ground state (0.91061) and the first excited state (1.69829) for the case $\mu = -\nu = \frac{1}{2}$, respectively. In all these cases we took $\gamma = m = 1, \lambda = 1.5$.

One more example we consider here is the case when we have spin-antisymmetric coupling in which S= -V, W=0. Taking R=0, and $V(y)=V_0 y\,;V_0=\lambda c$, we can write G(y) as:

$$G(y)=\frac{V_0}{\lambda}\left[\left(n+\tfrac{\mu+\nu+1}{2}\right)^2+\frac{\mu+\nu+1}{2}\right]y+\frac{m-\varepsilon}{2\lambda}+V_0\frac{\mu-\nu}{2\lambda}+\left(1-\tfrac{m+\varepsilon}{2\lambda}\right)\left(n+\tfrac{\mu+\nu+1}{2}\right)^2, \tag{3.2.19}$$

where $\mu^2=\nu^2=\dfrac{1}{4}$.

The J-matrix now takes the following form:

$$J_{n,m}=2\lambda\left\{\left[\frac{m-\varepsilon}{2\lambda}+\left(1-\tfrac{m+\varepsilon}{2\lambda}\right)\left(n+\tfrac{\mu+\nu+1}{2}\right)^2\right]\delta_{n,m}+\frac{V_0}{4\lambda}\left[2\left(n+\tfrac{\mu+\nu+1}{2}\right)^2-1\right]\left[\delta_{n,m-1}+\delta_{n,m+1}\right]\right\}, \tag{3.2.20}$$

The three-term recursion relation becomes:

$$\tfrac{\varepsilon}{2\lambda}f_n=a_n f_n+b_n\left[f_{n-1}+f_{n+1}\right], \tag{3.2.21}$$

where,

$$a_n=\left[\frac{\frac{m}{\lambda}-1+\left(1-\frac{m}{2\lambda}\right)\left(\left(n+\frac{\mu+\nu+1}{2}\right)^2+1\right)}{\left(n+\frac{\mu+\nu+1}{2}\right)^2+1}\right],$$

and,

$$b_n=\frac{V_0}{4\lambda}\left[\frac{2\left(n+\frac{\mu+\nu+1}{2}\right)^2-1}{\left(n+\frac{\mu+\nu+1}{2}\right)^2+1}\right],$$

We plot the energy spectrum versus the interaction potential for the four lowest energy states in the following figure:

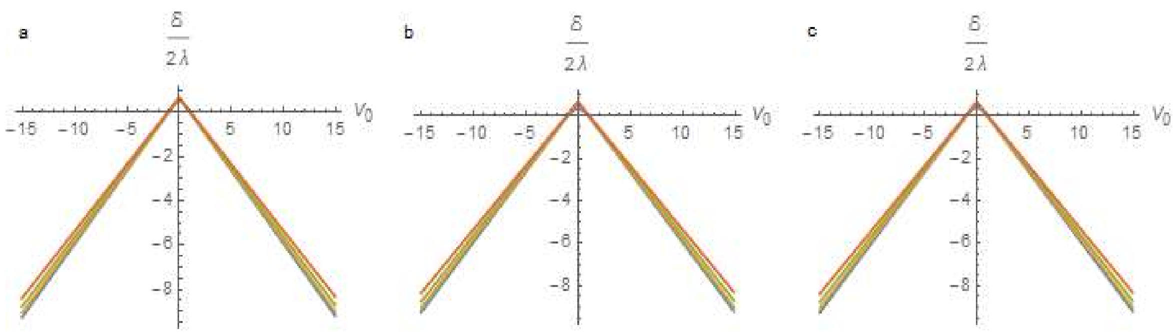

Figure 32: Plot of the four lowest energy states versus the potential strength V_0. (a) for the case where, $\mu = \nu = \frac{1}{2}$, (b) for the case where $\mu = \nu = -\frac{1}{2}$ and (c) for the case where $\mu = -\nu = \pm\frac{1}{2}$. Here we took m=1, λ=1.5.

We tabulated few polynomials for the recursion relation (3.2.21) in the table below:

Table 6: The first few polynomials of the recursion relation (3.2.21) for the case $\mu = \nu = \frac{1}{2}$. Here we took V_0=m=1, λ=1.5 and z=ε/2λ.

n	$f_n(z)$
1	$-6 + 12z$
2	$14.43 - 56.6z + 51.43z^2$
3	$-26.3 + 165.4z - 314.6z^2 + 181.5z^3$
4	$41.5 - 381.9z + 1162.6z^2 - 1421.7z^3 + 597.24z^4$

The next page shows a plot of the un-normalized spinor wavefunction components over the finite space of the Hamiltonian.

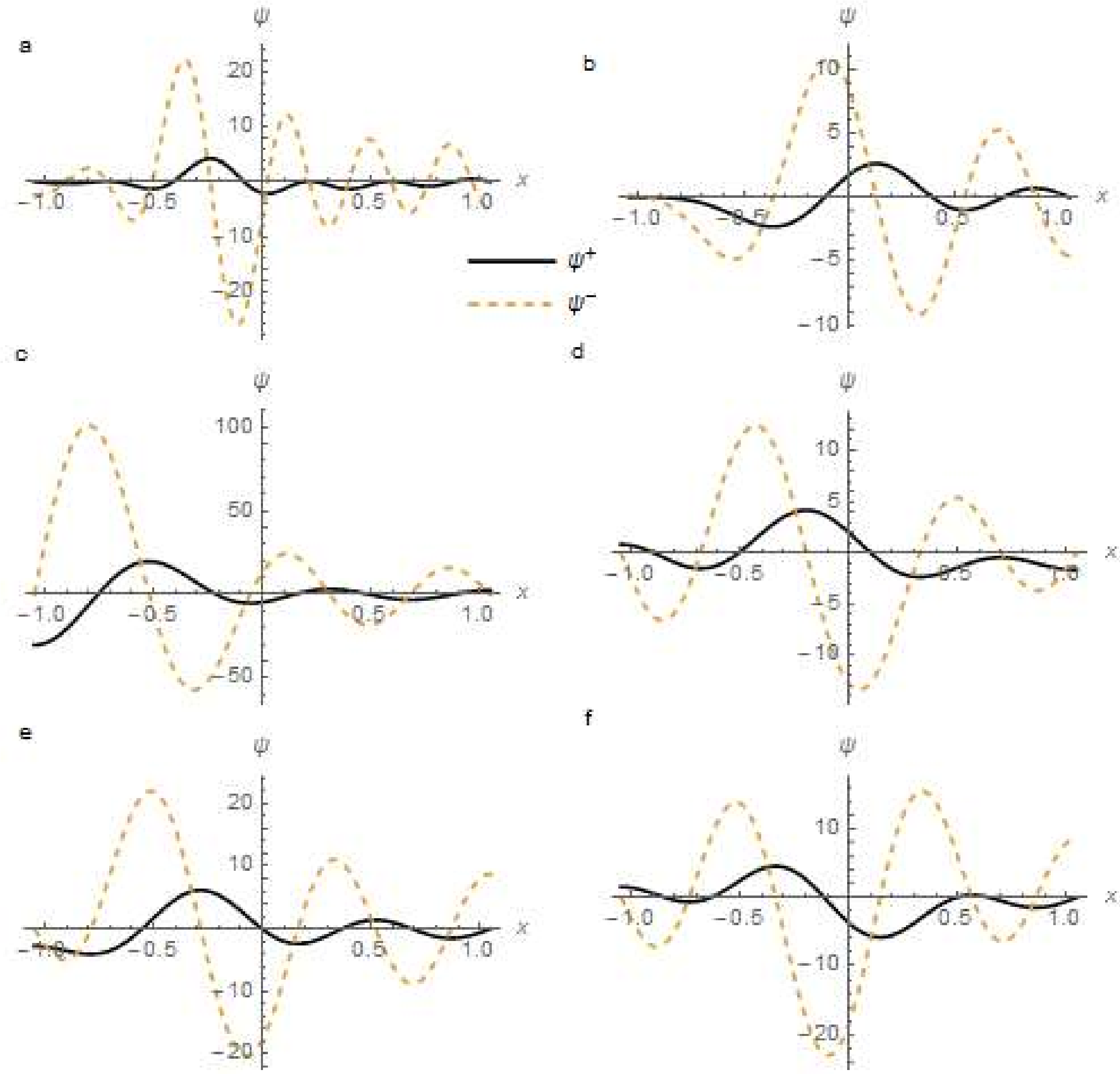

Figure 33: A plot of the un-normalized spinor wavefunction components for different choices of parameters. (a) and (b) for the ground state (1.30349) and the first excited state (1.77959) for the case when $\mu = \nu = \frac{1}{2}$, respectively. (c) and (d) are for the ground state (0.43597) and the first excited state (1.60674) for the case $\mu = \nu = -\frac{1}{2}$, respectively. (e) and (f) for the ground state (0.91061) and the first excited state (1.69829) for the case $\mu = -\nu = \frac{1}{2}$, respectively. In all these cases we took $\gamma = m = 1, \lambda = 1.5$.

We now move to the case of spin-symmetric coupling in which W is nonzero. Considering the sinusoidal transformation and taking $\mu^2 = \nu^2 = \frac{1}{4}$, we write G(y) as:

$$G(y) = \frac{2V + m - \varepsilon}{2\lambda} - c\left[\beta - \alpha - (\alpha + \beta)y + \frac{\sqrt{1-y^2}}{\lambda}R\right] + \left(1 - \tfrac{m+\varepsilon}{2\lambda}\right)\times \left\{\left(n + \tfrac{\mu+\nu+1}{2}\right)^2 + \frac{1}{\lambda^2}\left(R^2 - \lambda\sqrt{1-y^2}\dot{R}\right)\right\}, \tag{3.2.22}$$

To make G(y) linear, one would suggest the following constraints:

$$\begin{aligned} &\frac{1}{\lambda^2}\left[R^2(y) - \lambda\sqrt{1-y^2}\dot{R}\right] = \gamma y + \tau \\ &V - c\sqrt{1-y^2}R = \lambda(Ay + B) \end{aligned}, \tag{3.2.23}$$

Where $\{A, B, \gamma, \tau\}$ are constant parameters.

Selecting certain values of $\{A, B, \gamma, \tau\}$ such that we can solve the ODE for R(y) exactly. As a first example, we take $\gamma = \tau = B = 0$, this gives the following general solution of the ODE:

$$R(x) = -\frac{1}{x + \delta},$$

where δ is a real number chosen such that R(x) does not have singularities within the finite space of the Hamiltonian.

Thus, we can write the potentials in this case as follows:

$$V(x) = S(x) = -c\frac{\cos(\lambda x)}{x + \delta} + \lambda A \sin(\lambda x), \tag{3.2.24}$$

$$W(x)=-\left[\frac{1}{x+\delta}+c\lambda\cos(\lambda x)\right], \tag{3.2.25}$$

For this choice of potentials, the function G(y) is written as:

$$G(y)=\left[A+c\frac{\mu+\nu+1}{2}\right]y+\left(1-\tfrac{m+\varepsilon}{2\lambda}\right)\left[\left(n+\tfrac{\mu+\nu+1}{2}\right)^2+1\right]+\frac{m}{\lambda}-1+c\frac{\mu-\nu}{2}, \tag{3.2.26}$$

The J-matrix associated with this case reads as follows:

$$J_{n,m}=2\lambda\left\{\left[\left(1-\tfrac{m+\varepsilon}{2\lambda}\right)\left[\left(n+\tfrac{\mu+\nu+1}{2}\right)^2+1\right]+\frac{m}{\lambda}-1\right]\delta_{n,m}+\tfrac{2A-c}{4}\left[\delta_{n,m-1}+\delta_{n,m+1}\right]\right\}, \tag{3.2.27}$$

We plot the four lowest energy values versus the interaction parameter *A* as shown below:

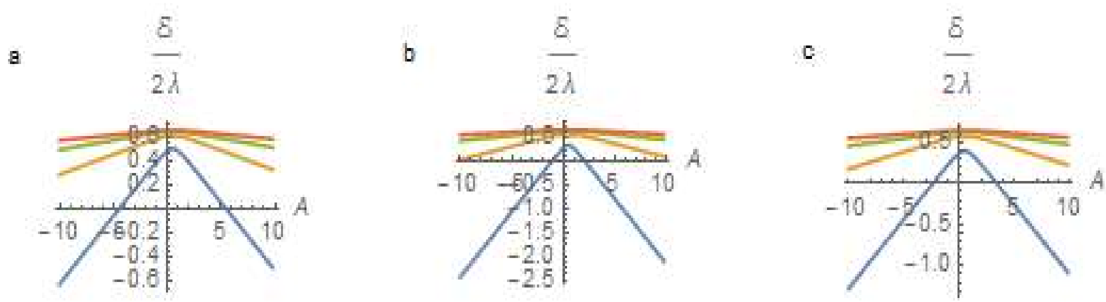

Figure 34: Plot of the four lowest energy states versus the interaction parameter A. (a) for the case where, $\mu=\nu=\frac{1}{2}$, (b) for the case where $\mu=\nu=-\frac{1}{2}$ and (c) for the case where $\mu=-\nu=\pm\frac{1}{2}$. Here we took m=1, λ=1.5 and c=1.

The expansion coefficients for this problem satisfy the following recursion relation:

$$\tfrac{\varepsilon}{2\lambda}f_n=\frac{\left(1-\tfrac{m}{2\lambda}\right)\left[\left(n+\tfrac{\mu+\nu+1}{2}\right)^2+1\right]+\frac{m}{\lambda}-1}{\left(n+\tfrac{\mu+\nu+1}{2}\right)^2+1}f_n+\frac{2A-c}{4\left(\left(n+\frac{\mu+\nu+1}{2}\right)^2+1\right)}\left[f_{n-1}+f_{n+1}\right], \tag{3.2.28}$$

The un-normalized spinor wavefunction is plotted for different choices of parameters in the following figure:

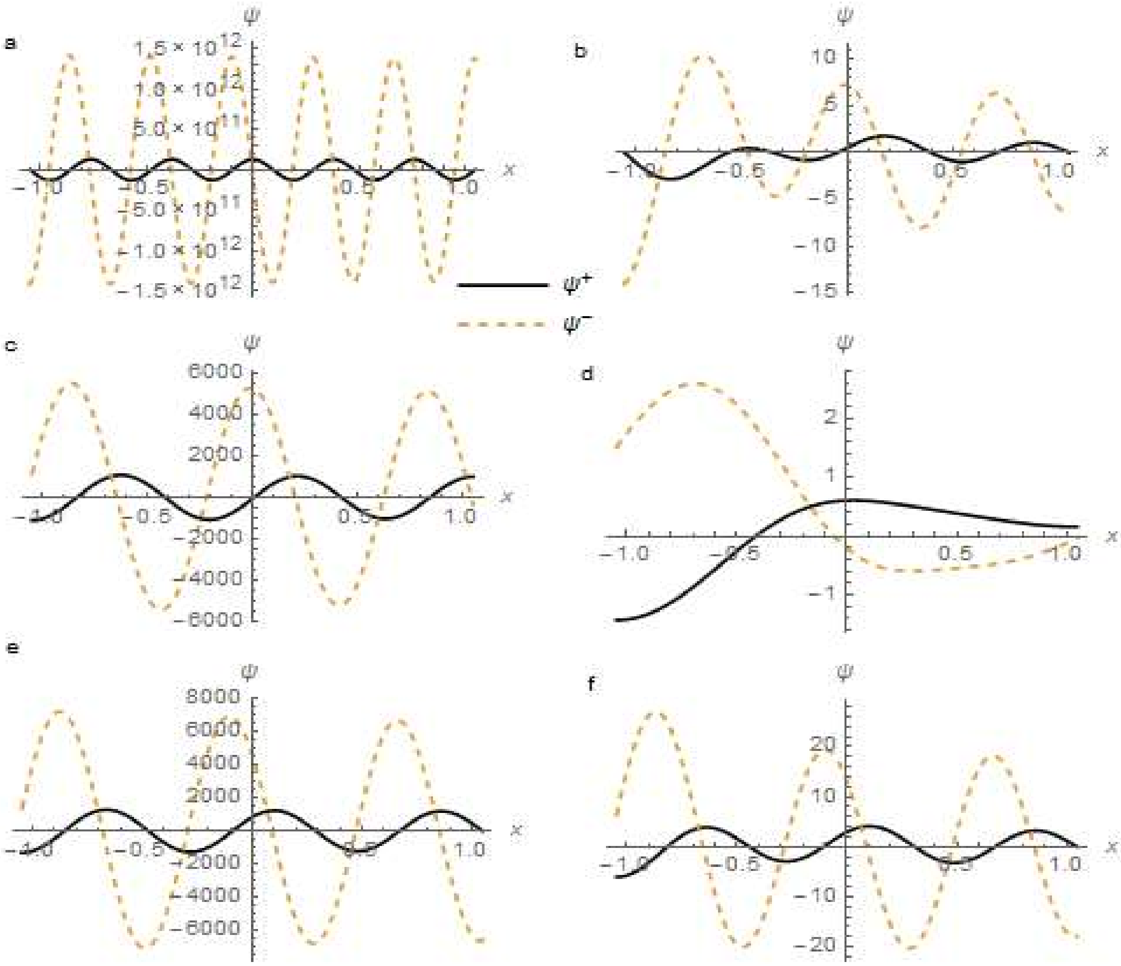

Figure 35: A plot of the un-normalized spinor wavefunction components for different choices of parameters. (a) and (b) for the ground state (1.43603) and the first excited state (1.80435) for the case when $\mu = \nu = \frac{1}{2}$, respectively. (c) and (d) are for the ground state (0.79420) and the first excited state (1.60466) for the case $\mu = \nu = -\frac{1}{2}$, respectively. (e) and (f) for the ground state (1.10678) and the first excited state (1.71461) for the case $\mu = -\nu = \frac{1}{2}$, respectively. In all these cases we took $A = c = m = 1$ and $\lambda = 1.5$.

Table 7: First four polynomial solutions of the recursion relation (3.2.28) where $z = \frac{\varepsilon}{2\lambda}$.

Here we took m=c=A=1, and λ=1.5.

n	$f_n(z)$
1	$-4+8z$
2	$47-176z+160z^2$
3	$-1186.67+6330.67z-11093.34z^2+6400z^3$
4	$52166.36-359066.67z+918432z^2-1035946.67z^3+435200z^4$

Another set of solvable potentials in this case can be obtained by taking $\gamma = B = 0, \tau = k^2$, where k is a real constant. This gives the following solution for R:

$$R(x) = -k\lambda \tanh\left[\lambda(kx-\delta)\right], \tag{3.2.29}$$

The solvable potentials are now written as follows:

$$V(x) = S(x) = \lambda\left\{-kc\cos(\lambda x)\tanh\left[\lambda(kx-\delta)\right] + A\sin(\lambda x)\right\}, \tag{3.2.30}$$

$$W(x) = -\lambda\left\{k\tanh\left[\lambda(kx-\delta)\right] + c\cos(\lambda x)\right\}, \tag{3.2.31}$$

Thus, the function G(y) associated with this case is written as follows:

$$G(y) = \left[A + c\frac{\mu+\nu+1}{2}\right]y + \left(1-\tfrac{m+\varepsilon}{2\lambda}\right)\times\left\{\left(n+\tfrac{\mu+\nu+1}{2}\right)^2 + k^2 + 1\right\} + \frac{m}{\lambda} - 1 + c\frac{\mu-\nu}{2}, \tag{3.2.32}$$

The J-matrix for this case becomes:

$$J_{n,m} = 2\lambda\left\{\left[\left(1-\tfrac{m+\varepsilon}{2\lambda}\right)\times\left\{\left(n+\tfrac{\mu+\nu+1}{2}\right)^2+k^2+1\right\}+\frac{m}{\lambda}-1\right]\delta_{n,m}+\tfrac{2A-c}{4}\left[\delta_{n,m-1}+\delta_{n,m+1}\right]\right\}, \quad (3.2.33)$$

The three-term recursion relation of the expansion coefficients is now written as:

$$\tfrac{\varepsilon}{2\lambda}f_n = \frac{\left(1-\frac{m}{2\lambda}\right)a_n+\dfrac{m}{\lambda}-1}{a_n}f_n+\tfrac{2A-c}{4a_n}\left[f_{n-1}+f_{n+1}\right], \quad (3.2.34)$$

where,

$$a_n = \left(n+\tfrac{\mu+\nu+1}{2}\right)^2+k^2+1.$$

We plot the spectrum versus k as shown below:

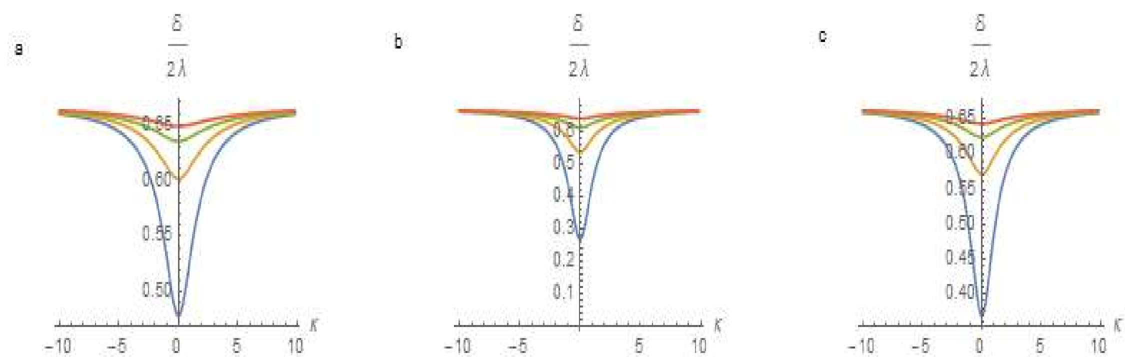

Figure 36: Plot of the four lowest energy states versus k. (a) for the case where, $\mu=\nu=\frac{1}{2}$, (b) for the case where $\mu=\nu=-\frac{1}{2}$ and (c) for the case where $\mu=-\nu=\pm\frac{1}{2}$. Here we took m=A=c=1 and λ=1.5.

The table below shows few polynomial solutions of the recursion relation:

Table 8: The first few polynomials of the recursion relation (3.2.34) for the case $\mu = \nu = \frac{1}{2}$. Here we took A=c=k=m=1, λ=1.5 and z=ε/2λ.

n	$f_n(z)$
1	$-6.67 + 12z$
2	$96.78 - 336z + 288z^2$
3	$-2703.1 + 13654.2z - 22848z^2 + 12672z^3$
4	$126048.41 - 831485.04z + 2049056z^2 - 2236416z^3 + 912384z^4$

The plot of the spinor wavefunction for different choice of parameters is shown in the next figure. One last example of solvable potentials for the spin-symmetric coupling cases we mention here is by obtained taking $\gamma = B = 0, \tau = -1$. This gives the following solution for R:

$$R(x) = \lambda \tan(\lambda x), \tag{3.2.35}$$

We write the solvable potentials as follows:

$$V(x) = S(x) = \lambda(A+c)\sin(\lambda x), \tag{3.2.36}$$

$$W(x) = \lambda\{\tan(\lambda x) - c\cos(\lambda x)\}, \tag{3.2.37}$$

The function G(y) for this case is:

$$G(y)=\left[A+c\frac{\mu+\nu+1}{2}\right]y+\left(1-\tfrac{m+\varepsilon}{2\lambda}\right)\left(n+\tfrac{\mu+\nu+1}{2}\right)^2+\frac{m}{\lambda}-1+c\frac{\mu-\nu}{2}, \quad (3.2.38)$$

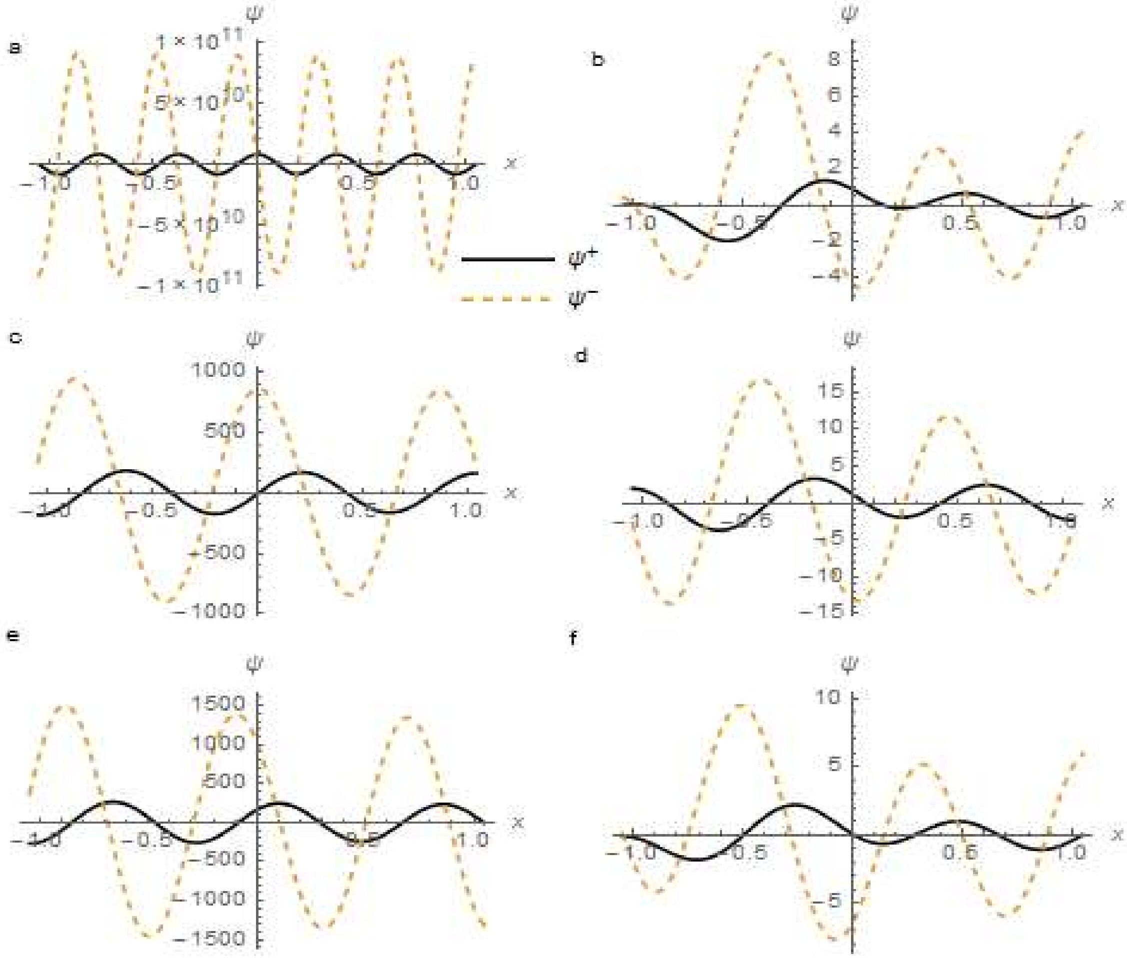

Figure 37: A plot of the un-normalized spinor wavefunction components for different choices of parameters. (a) and (b) for the ground state (1.59627) and the first excited state (1.83801) for the case when $\mu=\nu=\frac{1}{2}$, respectively. (c) and (d) are for the ground state (1.30907) and the first excited state (1.73059) for the case $\mu=\nu=-\frac{1}{2}$, respectively. (e) and (f) for the ground state (1.45004) and the first excited state (1.78599) for the case $\mu=-\nu=\frac{1}{2}$, respectively. In all these cases we took $A=c=k=m=1$ and $\lambda=1.5$.

Thus, the J-matrix for last case becomes:

$$J_{n,m} = 2\lambda\left\{\left[\left(1-\tfrac{m+\varepsilon}{2\lambda}\right)\left(n+\tfrac{\mu+\nu+1}{2}\right)^2+\frac{m}{\lambda}-1\right]\delta_{n,m}+\tfrac{2A-c}{4}\left[\delta_{n,m-1}+\delta_{n,m+1}\right]\right\}, \tag{3.2.39}$$

So, the three-term recursion relation will read as follows:

$$\tfrac{\varepsilon}{2\lambda} f_n = \frac{\left(1-\tfrac{m}{2\lambda}\right)a_n+\dfrac{m}{\lambda}-1}{a_n} f_n + \tfrac{2A-c}{4a_n}\left[f_{n-1}+f_{n+1}\right], \tag{3.2.40}$$

Where: $a_n = \left(n+\tfrac{\mu+\nu+1}{2}\right)^2$.

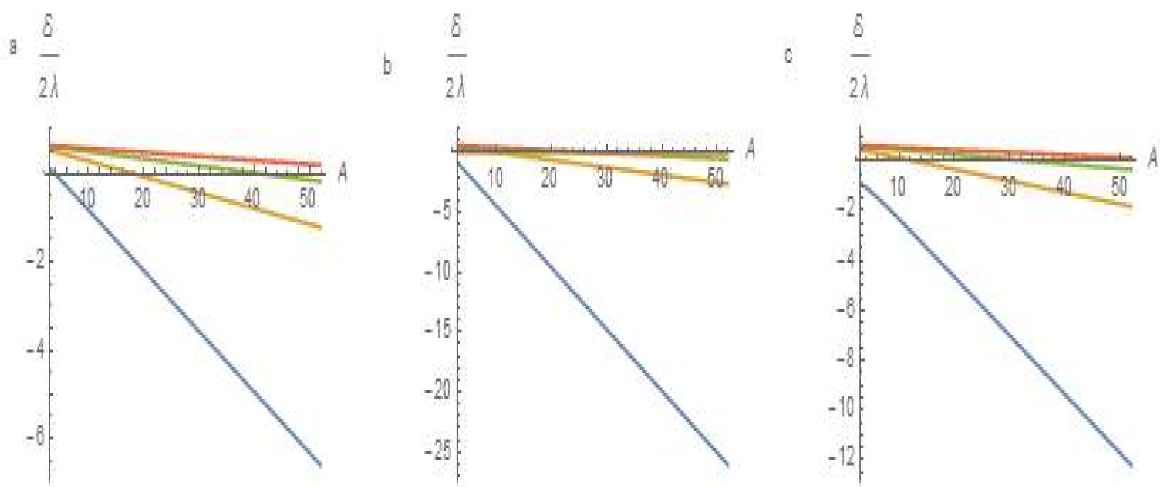

Figure 38: Plot of the four lowest energy states versus A. (a) for the case where, $\mu = \nu = \frac{1}{2}$, (b) for the case where $\mu = \nu = -\frac{1}{2}$ and (c) for the case where $\mu = -\nu = \pm\frac{1}{2}$. Here we took m=c=1 and λ=1.5.

The table below shows few polynomial solutions of the recursion relation (3.2.40) for special choice of parameters indicated in table 9.

Table 9: The first few polynomials of the recursion relation (3.2.40) for the case $\mu = \nu = \frac{1}{2}$. Here we took A=c=m=1, λ=1.5 and z=ε/2λ.

n	$f_n(z)$
1	$-1.34 + 4z$
2	$11.45 - 58.67z + 64z^2$
3	$-258.1 + 1737.78z - 3562.67z^2 + 2304z^3$
4	$10655.62 - 88286.23z + 258410.67z^2 - 323242.67z^3 + 147456z^4$

The plot of the spinor wavefunction for different values of the parameters for both the ground state and the first excited state are shown in Figure 36.

In summary, we have considered different coupling conditions and in each case we have shown the solvable potentials that can be treated within our approach. For each potential set we plotted the potentials and the energy spectra versus the potential strength. Also, we listed few of the polynomial solutions of each recursion relation and then plotted the spinor wavefunction components for different parameters for both ground state and the first excited state. Finally, we should clarify here that this is not the end of the story, there are other solvable potentials that belongs to Jacobi basis but our goal here is not to list all possible results but rather showed a good set of examples to show the strength of our approach.

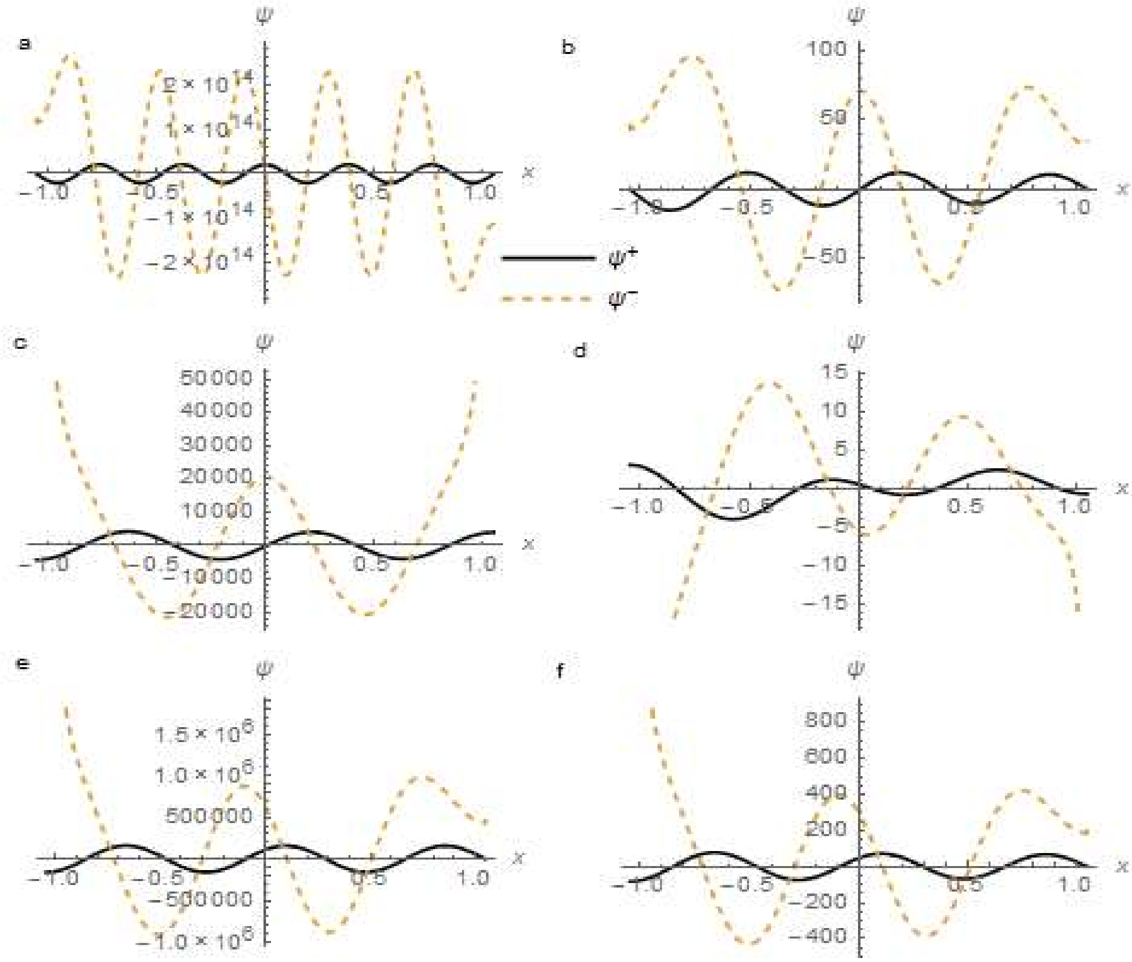

Figure 39: A plot of the un-normalized spinor wavefunction components for different choices of parameters. (a) and (b) for the ground state (0.95534) and the first excited state (1.74478) for the case when $\mu = \nu = \frac{1}{2}$, respectively. (c) and (d) are for the ground state (0.52922) and the first excited state (1.78667) for the case $\mu = \nu = -\frac{1}{2}$, respectively. (e) and (f) for the ground state (-2.03101) and the first excited state (1.53771) for the case $\mu = -\nu = \frac{1}{2}$, respectively. In all these cases we took $A = c = m = 1$ and $\lambda = 1.5$.

CHAPTER 4

Conclusions and Future Work

In this work we have succeeded in treating the 1D Dirac equation in the tridiagonal matrix representation approach (TMRA). This important requirement resulted in a large set of solvable potentials, most of which fall outside the usual class of solvable potentials known in literature. In this approach we choose the wavefunction to be expandable in a convergent series with respect to square integrable basis functions and require the wave operator to be tridiagonal and symmetric, this gives a wider class of solvable potentials. In this work we have considered the general approach and then considered cases in which we got a rich class of solvable potentials. We restricted our work to Laguerre and Jacobi bases but one might try other square integrable basis such as Hermite basis. For each potential we calculated the wavefunction and we wrote down the three-term recursion relation for the expansion coefficients for each case. Unfortunately, we have solved only two of these recursion relations by direct comparison with Mixner-Pollaczek orthogonal polynomials. For other recursion relations, associated with non-classical orthogonal polynomials, we used Mathematical to generate exact solutions at any order. It should be noticed here that the wavefunction series converges very fast which means that we do not have to worry about the existence of exact closed form solutions of the three-term recursion relations. For future work we should clarify here that the problem still open and other solvable potentials can be found using our method either by using different basis, say, Hermite basis, or by using different treatments within the same bases that we used. Moreover, we are interested in applying our approach to Dirac equation in higher dimensions. Finally, we are interested in implementing our results in some real physical applications like graphene.

Appendices

Appendix A: Special Functions

A.1: Jacobi Polynomials

Jacobi polynomials denoted by $P_n^{(\mu,\nu)}(y)$, are one special class of classical orthogonal polynomials defined on $y \in [-1,1]$. One can use the Rodrigues' formula to calculate these polynomials which is given below [15, 16]:

$$P_n^{(\mu,\nu)}(y) = \frac{(-1)^n}{2^n n!}(1-y)^{-\mu}(1+y)^{-\nu}\frac{d^n}{dy^n}\left\{(1-y)^{\mu}(1+y)^{\nu}\left(1-y^2\right)^n\right\}, \tag{A1}$$

These polynomials satisfies the following useful properties:

$$\left\{\left(1-y^2\right)\frac{d^2}{dy^2} - \left[(\mu+\nu+2)y+\mu-\nu\right]\frac{d}{dy} + n(n+\mu+\nu+1)\right\}P_n^{(\mu,\nu)}(y) = 0, \tag{A2}$$

$$\left(1-y^2\right)\frac{dP_n^{(\mu,\nu)}}{dy} = -n\left(y+\frac{\nu-\mu}{2n+\nu+\mu}\right)P_n^{(\mu,\nu)} + 2\frac{(n+\nu)(n+\mu)}{2n+\mu+\nu}P_n^{(\mu,\nu)}, \tag{A3}$$

$$\begin{aligned} yP_n^{(\mu,\nu)}(y) = {} & \frac{\nu^2-\mu^2}{(2n+\mu+\nu)(2n+\mu+\nu+2)}P_n^{(\mu,\nu)}(y) + \frac{2(n+\nu)(n+\mu)}{(2n+\mu+\nu)(2n+\mu+\nu+1)}P_{n-1}^{(\mu,\nu)}(y) \\ & + \frac{2(n+1)(n+\mu+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+2)}P_{n+1}^{(\mu,\nu)}(y) \end{aligned} \tag{A4}$$

$$\int_{-1}^{1}(1-y)^{\mu}(1+y)^{\nu}P_n^{(\mu,\nu)}P_m^{(\mu,\nu)}dy = \frac{2^{\mu+\nu+1}}{2n+\mu+\nu+1}\frac{\Gamma(n+\mu+1)\Gamma(n+\nu+1)}{\Gamma(n+\mu+\nu+1)n!}\delta_{n,m}, \tag{A5}$$

A.2: Associated Laguerre Polynomials

The associated Laguerre polynomials $L_n^\nu(y)$ are solutions to the following second order differential equation [15, 16]:

$$\left\{ y\frac{d^2}{dy^2} + (\nu+1-y)\frac{d}{dy} + n \right\} L_n^\nu(y) = 0 \tag{A6}$$

where n is a non-negative integer and ν is a real number. These polynomials satisfy the following properties:

$$y\frac{d}{dy}L_n^\nu(y) = nL_n^\nu(y) - (n+\nu)L_{n-1}^\nu(y)\,, \tag{A7}$$

$$yL_n^\nu(y) = (2n+\nu+1)L_n^\nu(y) - (n+\nu)L_{n-1}^\nu(y) - (n+1)L_{n+1}^\nu(y)\,, \tag{A8}$$

$$\int_0^\infty y^\nu e^{-y} L_n^\nu(y) L_m^\nu(y) dy = \frac{\Gamma(n+\nu+1)}{n!}\delta_{n,m}\,, \tag{A9}$$

where $\Gamma(n+\nu+1)$ is the Gamma function.

One can find these polynomials at any order using the following Rodrigues formula:

$$L_n^\nu(y) = \frac{y^{-\nu}}{n!}\left(\frac{d}{dy} - 1\right)^n y^{n+\nu} \tag{A10}$$

or using the generating function given below:

$$\sum_n t^n L_n^\nu(y) = \frac{1}{(1-t)^{\nu+1}} e^{-\frac{ty}{1-y}} \tag{A11}$$

A.3: Bessel Functions

Bessel polynomials are solutions to Bessel differential equation reads as:

$$x^2 y''(x) + xy'(x) + (x^2 - p^2) y(x) = 0 \quad , \tag{A12}$$

Where p is an arbitrary constant.

The general solution to this ODE is given by:

$$y(x) = AJ_p(x) + BY_p(x), \tag{A13}$$

Here $J_p(x)$ and $Y_p(x)$ are called Bessel functions of 1st and 2nd kinds, respectively. These functions are defined in the following compact forms:

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n+p}, \tag{A14}$$

$$Y_p(x) = \frac{Cos(\pi p) J_p(x) - J_{-p}(x)}{Sin(\pi p)}, \tag{A15}$$

It is clear that Eq. (A15) is true for none-integer p.

Here are some useful properties of $J_p(x)$ [15-18]:

$$\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x), \tag{A16}$$

$$\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x), \tag{A17}$$

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x), \tag{A18}$$

$$J_{p-1}(x) - J_{p+1}(x) = 2\frac{d}{dx} J_p(x), \tag{A19}$$

A.4 : Meixner-Pollaczek Polynomials

These polynomials are defined in terms of the hypergeometric function as follows [17]:

$$P_n^{\mu}\left(y;\theta\right)=\sqrt{\frac{\Gamma\left(n+2\mu\right)}{\Gamma\left(2\mu\right)\Gamma\left(n+1\right)}}e^{in\theta}\,{}_2F_1\left(-n,\mu+iy;2\mu;1-e^{-2i\theta}\right), \tag{A20}$$

where $y\in\left]-\infty,\infty\right[$, $\mu>0$, and $0<\theta<\pi$.

These polynomials satisfy the following three-term recursion relation:

$$\begin{aligned}\left[z\sin\left(\theta\right)\right]P_n^{\mu}\left(z;\theta\right)=&-\left[\left(n+\mu\right)\cos\left(\theta\right)\right]P_n^{\mu}\left(z;\theta\right)\\&+\frac{1}{2}\left[\sqrt{n\left(n+2\mu-1\right)}P_{n-1}^{\mu}\left(z;\theta\right)+\sqrt{\left(n+1\right)\left(n+2\mu\right)}P_{n+1}^{\mu}\left(z;\theta\right)\right],\end{aligned} \tag{A21}$$

The associated weight function reads as follows:

$$\omega\left(z\right)=\frac{1}{2\pi\,\Gamma\left(2\mu\right)}\left(2\sin\theta\right)^{2\mu}e^{(2\theta-\pi)z}\,|\,\Gamma\left(\mu+iz\right)|^{2}, \tag{A22}$$

One can generate these polynomials using the following generating function:

$$\sum_{n=0}^{\infty}\sqrt{\frac{\Gamma\left(n+2\mu\right)}{\Gamma\left(2\mu\right)\Gamma\left(n+1\right)}}P_n^{\mu}\left(y;\theta\right)t^n=\left(1-te^{i\theta}\right)^{-\mu+iy}\left(1-te^{-i\theta}\right)^{-\mu-iy}, \tag{A23}$$

To calculate the bound states we use the infinite spectrum formula associated with these polynomials which is given below:

$$z^2=-\left(n+\mu\right)^2, \tag{A24}$$

In some cases we get recursion relation similar to that in (A21) but we cannot have the sine and the cosine between -1 and 1. In such circumstances we transform the problem by making a replacement θ→ iθ which makes (A21) reads as:

$$\begin{aligned}\left[iz\sinh(\theta)\right]P_n^{\mu}(z;\theta)&=-\left[(n+\mu)\cosh(\theta)\right]P_n^{\mu}(z;\theta)\\&+\frac{1}{2}\left[\sqrt{n(n+2\mu-1)}P_{n-1}^{\mu}(z;\theta)+\sqrt{(n+1)(n+2\mu)}P_{n+1}^{\mu}(z;\theta)\right],\end{aligned}\qquad \text{(A25)}$$

Solutions to (A25) are called the discrete Meixner polynomials which are written as:

$$M_n^{2\mu}(m;\beta)=\sqrt{\frac{\Gamma(n+2\mu)\beta^n}{\Gamma(2\mu)\Gamma(n+1)}}\,{}_2F_1\left(-n,-m;2\mu;1-\beta^{-1}\right),\qquad \text{(A26)}$$

where: $\beta=e^{-2\theta}$ and m stands for the mth bound state.

However, the orthogonality relation of these polynomials is given below:

$$\sum_{m=0}^{\infty}\omega_m^{\alpha}(\beta)M_n^{\alpha}(m;\beta)M_{n'}^{\alpha}(m;\beta)=\delta_{n,n'},\qquad \text{(A27)}$$

Where $\omega_m^{\alpha}(\beta)$ is the weight function associated with these polynomials which reads as:

$$\omega_m^{\alpha}(\beta)=(1-\beta)^{\alpha}\frac{\Gamma(m+\alpha)\beta^m}{\Gamma(\alpha)\Gamma(m+1)},\qquad \text{(A28)}$$

Appendix B: Mathematical Derivations

B.1: Eigenvalue Form of Dirac Equation

We begin here by introducing the most general form of 1+1 (space-time) Dirac equation:

$$\left\{\gamma^{\mu}\left[i\partial_{\mu}-A_{\mu}\left(x\right)\right]-I_{2}S\left(x\right)-\gamma^{5}W\left(x\right)\right\}\left|\Psi(x,t)\right\rangle=m\left|\Psi(x,t)\right\rangle, \tag{B1}$$

Where: $\gamma^{0}=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\gamma^{1}=\begin{pmatrix}0&-i\\-i&0\end{pmatrix}\&\gamma^{5}=i\gamma^{0}\gamma^{1}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$. Recalling that the vector potential is $A_{\mu}=(V,U)^{T}$ and $\partial_{\mu}=\dfrac{\partial}{dx^{\mu}}$. In this case, $\partial_{0}=\dfrac{\partial}{\partial t},\partial_{1}=\dfrac{\partial}{\partial x}$.

Expanding the spinor as $\left|\Psi(x,t)\right\rangle=e^{-\varepsilon t}\left|\psi\left(x\right)\right\rangle=e^{-\varepsilon t}\left(\psi^{+}\left(x\right),\psi^{-}\left(x\right)\right)^{T}$, and using all the above details, we can rewrite Eq. (B1) in the following matrix form:

$$\begin{pmatrix}m+S\left(x\right)+V\left(x\right) & -\dfrac{d}{dx}+W-iU\left(x\right)\\ \dfrac{d}{dx}+W+iU\left(x\right) & -m-S\left(x\right)+V\left(x\right)\end{pmatrix}\begin{pmatrix}\psi^{+}(x)\\ \psi^{-}(x)\end{pmatrix}=\varepsilon\begin{pmatrix}\psi^{+}(x)\\ \psi^{-}(x)\end{pmatrix}, \tag{B2}$$

A proper choice of gauge gives an irrelevant phase factor in the spinor wavefunction. If we consider here $\psi^{\pm}(x)\rightarrow e^{-i\Lambda(x)}\psi^{\pm}(x)$ provided that $U=d\Lambda/dx$, this always kills the time component of the vector potential, that is, U =0. Thus, Eq. (B2) takes the following final form:

$$\begin{pmatrix}m+S\left(x\right)+V\left(x\right) & -\dfrac{d}{dx}+W\\ \dfrac{d}{dx}+W & -m-S\left(x\right)+V\left(x\right)\end{pmatrix}\begin{pmatrix}\psi^{+}(x)\\ \psi^{-}(x)\end{pmatrix}=\varepsilon\begin{pmatrix}\psi^{+}(x)\\ \psi^{-}(x)\end{pmatrix}, \tag{B3}$$

B.2: The J-matrix in a General Basis

The J-matrix of the Dirac equation (B3) can be written as:

$$J_{n,m} = \langle \phi_n | (H-\varepsilon) | \phi_m \rangle = \langle \phi_n^+ | Q_+ | \phi_m^+ \rangle + \langle \phi_n^- | Q_- | \phi_m^- \rangle + \langle \phi_n^+ | \left(-\frac{d}{dx} + W \right) | \phi_m^- \rangle + \langle \phi_n^- | \left(\frac{d}{dx} + W \right) | \phi_m^+ \rangle ,$$

(B4)

Where $Q_{\pm} = V \pm S \pm m - \varepsilon$.

The upper and the lower components of the basis spinor, ϕ_n^+ and ϕ_n^-, are related through the kinetic balance relation [13]:

$$\phi_n^-(x) = \frac{1}{\lambda}\left[\frac{d}{dx} + R(x) \right] \phi_n^+(x) = \frac{1}{\lambda}\left[y' \frac{d}{dy} + R(y) \right] \phi_n^+(y) , \tag{B5}$$

Where λ is just a scale factor and R(y) is an arbitrary function of y which will be determined later such that we achieve the tridiagonal constraints.

Using Eq. (B5), we can simplify Eq. (B4) by eliminating the lower component of the spinor. The first term is already in terms of the upper component only, so we begin with the second term in Eq. (B4) as follows:

$$\begin{aligned} \langle \phi_n^- | Q_- | \phi_m^- \rangle &= \frac{1}{\lambda^2} \left\langle \left[y' \frac{d}{dy} + R(y) \right] \phi_n^+(y) \middle| Q_- \middle| \left[y' \frac{d}{dy} + R(y) \right] \phi_m^+(y) \right\rangle \\ &= \frac{1}{\lambda^2} \left\{ \left\langle y' \frac{d\phi_n^+}{dy} \middle| Q_- \middle| y' \frac{d\phi_m^+}{dy} \right\rangle + \left\langle y' \frac{d\phi_n^+}{dy} \middle| RQ_- \middle| \phi_m^+ \right\rangle + \left\langle \phi_n^+ \middle| RQ_- y' \frac{d}{dy} \middle| \phi_m^+ \right\rangle \right. \\ &\quad \left. + \left\langle \phi_n^+ \middle| R^2 Q_- \middle| \phi_m^+ \right\rangle \right\} \end{aligned} \tag{B6}$$

The first two terms in the second line of Eq. (B6) need integration by parts to remove the derivatives over ϕ_n^+. We begin with the first term:

$$\begin{aligned}\frac{1}{\lambda^2}\left\langle y'\frac{d\phi_n^+}{dy}\right|Q_-\left|y'\frac{d\phi_m^+}{dy}\right\rangle &= \frac{1}{\lambda^2}\int_{x_-}^{x_+}\frac{d\phi_n^+}{dx}Q_-\frac{d\phi_m^+}{dx}dx \\ &= \frac{1}{\lambda^2}\left\{Q_-\phi_n^+\frac{d\phi_m^+}{dx}\Big|_{x=x_-}^{x=x_+} - \int_{x_-}^{x_+}\phi_n^+\frac{d}{dx}\left[Q_-\frac{d\phi_m^+}{dx}\right]dx\right\}\end{aligned}$$

Imposing that the basis vanishes at the boundaries gives:

$$\begin{aligned}\frac{1}{\lambda^2}\left\langle y'\frac{d\phi_n^+}{dy}\right|Q_-\left|y'\frac{d\phi_m^+}{dy}\right\rangle &= -\frac{1}{\lambda^2}\int_{x_-}^{x_+}\phi_n^+\frac{d}{dx}\left[Q_-\frac{d\phi_m^+}{dx}\right]dx \\ &= \frac{1}{\lambda^2}\left\langle\phi_n^+\right|\left[-Q_-\frac{d^2}{dx^2}-\frac{dQ_-}{dx}\frac{d}{dx}\right]\left|\phi_m^+\right\rangle\end{aligned}$$

Making a transformation $x \to y(x)$, we get:

$$\frac{1}{\lambda^2}\left\langle y'\frac{d\phi_n^+}{dy}\right|Q_-\left|y'\frac{d\phi_m^+}{dy}\right\rangle = \frac{1}{\lambda^2}\left\langle\phi_n^+\right|\left[-y'^2Q_-\frac{d^2}{dx^2}-\left(y''Q_-+y'^2\dot{Q}_-\right)\frac{d}{dy}\right]\left|\phi_m^+\right\rangle \qquad \text{(B7)}$$

Similarly, the second term in Eq. (B6) becomes:

$$\begin{aligned}\left\langle y'\frac{d\phi_n^+}{dy}\right|RQ_-\left|\phi_m^+\right\rangle &= \int_{x_-}^{x_+}\frac{d\phi_n^+}{dx}RQ_-\phi_m^+dx \\ &= RQ_-\phi_n^+\phi_m^+\Big|_{x=x_-}^{x=x_+} - \int_{x_-}^{x_+}\phi_n^+\frac{d}{dx}\left[RQ_-\phi_m^+\right]dx \qquad , \\ &= \left\langle\phi_n^+\right|\left(-y'RQ_-\frac{d}{dy}-y'\left(\dot{R}Q_-+R\dot{Q}_-\right)\right)\left|\phi_m^+\right\rangle\end{aligned} \qquad \text{(B8)}$$

Using Eq. (B7) and Eq. (B8), we can simplify Eq. (B6) to the following form:

$$\left\langle \phi_n^- \middle| Q_- \middle| \phi_m^- \right\rangle = \frac{1}{\lambda^2} \left\langle \phi_n^+ \middle| Q_- \left\{ -y'^2 \frac{d^2}{dy^2} - \left(y'' + y'^2 \frac{\dot{Q}_-}{Q_-} \right) \frac{d}{dy} - y' \left(\dot{R} + R \frac{\dot{Q}_-}{Q_-} \right) + R^2 \right\} \middle| \phi_m^+ \right\rangle, \tag{B9}$$

Notice here that this term is symmetric since $\left\langle \phi_n^- \middle| Q_- \middle| \phi_m^- \right\rangle = \left\langle \phi_m^- \middle| Q_- \middle| \phi_n^- \right\rangle$.

The last term in Eq. (B4) is equivalent to the third term but with $n \leftrightarrow m$:

$$\begin{aligned} \left\langle \phi_n^- \middle| \left(\frac{d}{dx} + W \right) \middle| \phi_m^+ \right\rangle &= \left\langle \phi_n^- \middle| \frac{d}{dx} \middle| \phi_m^+ \right\rangle + \left\langle \phi_n^- \middle| W \middle| \phi_m^+ \right\rangle \\ &= \int_{x_-}^{x_+} \phi_n^- \frac{d\phi_m^+}{dx} dx + \left\langle \phi_n^- \middle| W \middle| \phi_m^+ \right\rangle \\ &= -\int_{x_-}^{x_+} \phi_m^+ \frac{d\phi_n^-}{dx} dx + \left\langle \phi_n^- \middle| W \middle| \phi_m^+ \right\rangle \\ &= \left\langle \phi_m^+ \middle| \left(-\frac{d}{dx} + W \right) \middle| \phi_n^- \right\rangle \end{aligned} \tag{B10}$$

This brings a symmetry to the problem and one can work out one of the last terms in Eq. (B4), say the third term:

$$\begin{aligned} \left\langle \phi_n^+ \middle| \left(-\frac{d}{dx} + W \right) \middle| \phi_m^- \right\rangle &= \frac{1}{\lambda} \left\langle \phi_n^+ \middle| \left(-\frac{d}{dx} + W \right) \left(\frac{d}{dx} + R \right) \middle| \phi_m^+ \right\rangle \\ &= \frac{1}{\lambda} \left\langle \phi_n^+ \middle| \left\{ -y'^2 \frac{d^2}{dy^2} - \left(y'' + y'R - y'W \right) \frac{d}{dy} - y'\dot{R} + WR \right\} \middle| \phi_m^+ \right\rangle \end{aligned} \tag{B11}$$

Collecting all of these results back in Eq. (B4) and considering the symmetry under index exchange, we can write the J-matrix in this very compact form:

$$\begin{aligned} J_{n,m} = \frac{-1}{\lambda} \left\langle \phi_n^+ \middle| \left\{ \left(2 + \lambda^{-1} Q_- \right) \left[y'^2 \frac{d^2}{dy^2} + y'' \frac{d}{dy} \right] + \lambda^{-1} y' V_-' \frac{d}{dy} \right\} \middle| \phi_m^+ \right\rangle \\ + \frac{1}{\lambda} \left\langle \phi_n^+ \middle| \left[\lambda^{-1} Q_- \left(R^2 - R' \right) + R \left(2W - \lambda^{-1} V_-' \right) - R' - W' + \lambda Q_+ \right] \middle| \phi_m^+ \right\rangle \end{aligned}, \tag{B12}$$

Notice here the dot over the variable stands for the derivative with respect to y.

B.3: The J-matrix in Jacobi Basis

We start by expanding the upper component of the spinor in Jacobi basis as follows:

$$\phi_n^+(y) = A_n (1-y)^\alpha (1+y)^\beta P_n^{(\mu,\nu)}(y), \tag{B13}$$

The first two derivatives of (B13) are given below:

$$\frac{d}{dy}\phi_n^+ = A_n (1-y)^\alpha (1+y)^\beta \left\{ \frac{\beta}{1+y} - \frac{\alpha}{1-y} + \frac{d}{dy} \right\} P_n^{(\mu,\nu)}, \tag{B14}$$

$$\frac{d^2}{dy^2}\phi_n^+ = A_n (1-y)^\alpha (1+y)^\beta \left\{ \frac{\beta(\beta-1)}{(1+y)^2} + \frac{\alpha(\alpha-1)}{(1-y)^2} - \frac{2\alpha\beta}{1-y^2} + 2\left(\frac{\beta}{1+y} - \frac{\alpha}{1-y} \right)\frac{d}{dy} + \frac{d^2}{dy^2} \right\} P_n^{(\mu,\nu)}, \tag{B15}$$

Using these derivatives in Eq. (B12), we can write the J-matrix in this basis as:

$$\begin{aligned} J_{n,m} = &-\frac{A_n A_m}{\lambda} \left\langle P_n^{(\mu,\nu)} \right| (1-y)^{2\alpha} (1+y)^{2\beta} \left(1 + \tfrac{Q}{2\lambda} \right) y'^2 \left[\frac{d^2}{dy^2} + \left(\tfrac{2\beta+b}{1+y} - \tfrac{2\alpha+a}{1-y} \right) \frac{d}{dy} \right] \left| P_m^{(\mu,\nu)} \right\rangle \\ &-\frac{A_n A_m}{\lambda} \left\langle P_n^{(\mu,\nu)} \right| (1-y)^{2\alpha} (1+y)^{2\beta} y' \left(R - W + y' \tfrac{\dot{Q}}{2\lambda} \right) \frac{d}{dy} \left| P_m^{(\mu,\nu)} \right\rangle \\ &+\frac{A_n A_m}{\lambda} \left\langle P_n^{(\mu,\nu)} \right| (1-y)^{2\alpha} (1+y)^{2\beta} F(y) \left| P_m^{(\mu,\nu)} \right\rangle + n \leftrightarrow m \end{aligned} \tag{B16}$$

Where the coordinate transformation taken to be $y' = \lambda (1-y)^a (1+y)^b$ which gives $y'' = y'^2 \left[\frac{b}{1+y} - \frac{a}{1-y} \right]$ for real parameters a and b. The function F(y) is given below:

$$F(y)=\frac{\lambda Q_+}{2}-\left(R-W+y'\frac{\dot{Q}_-}{2\lambda}\right)\left[R+y'\left(\frac{\beta}{1+y}-\frac{\alpha}{1-y}\right)\right]+\left(1+\frac{Q_-}{2\lambda}\right)\times$$
$$\left\{R^2-y'\dot{R}-y'^2\left[\frac{\alpha(\alpha+a-1)}{(1-y)^2}+\frac{\beta(\beta+b-1)}{(1+y)^2}-\frac{\alpha b+\beta a+2\alpha\beta}{1-y^2}\right]\right\}, \tag{B17}$$

The next step here is to make use of the properties of the Jacobi polynomials such that we makes the wave operator tridiagonal and symmetric. One should be careful here about the integration measure since we are dealing with new variable y, all integrations will transform as: $\int f(...)g dx = \int f(...)g\frac{dy}{y'}$. By comparing the first line in (B16) with the differential equation of these polynomials (A2) and the orthogonality relation (A5) we get:

$$2\beta+b=\nu+1 \;\&\; 2\alpha+a=\mu+1, \tag{B18}$$

Comparing the second line in (B16) with the orthogonality relation (A5), we impose:

$$R-W+y'\frac{\dot{Q}_-}{2\lambda}=cy', \tag{B19}$$

Using the constraints in (B18) and (B19), we can write Eq. (B16) in the following simple form:

$$J_{n,m}=2c\lambda\,\Omega_{n,m}+2\lambda A_n A_m\int_{-1}^{+1}(1-y)^{\mu}(1+y)^{\nu}G(y)P_n^{(\mu,\nu)}(y)P_m^{(\mu,\nu)}(y)dy, \tag{B20}$$

where,

$$G(y)=\frac{\lambda Q_+}{2}\frac{1-y^2}{y'^2}-c\left[\beta-\alpha-(\alpha+\beta)y+\frac{1-y^2}{y'}R\right]+\left(1+\frac{Q_-}{2\lambda}\right)\left\{\left(n+\frac{\mu+\nu+1}{2}\right)^2\right.$$
$$\left.-\frac{1}{4}\left(\mu^2+\nu^2+2ab-1\right)-\frac{1}{4}\left[\mu^2-(a-1)^2\right]\frac{1+y}{1-y}-\frac{1}{4}\left[\nu^2-(b-1)^2\right]\frac{1-y}{1+y}+\frac{1-y^2}{y'^2}\left(R^2-y'\dot{R}\right)\right\}, \tag{B21}$$

and,

$$\Omega_{n,m}=\frac{2n(\nu-\mu)(n+\mu+\nu+1)}{(2n+\mu+\nu)(2n+\mu+\nu+2)}\delta_{n,m}-(\mu+\nu+2)\left[\frac{1}{2n+\mu+\nu}\sqrt{\frac{n(n+\mu)(n+\nu)(n+\mu+\nu)}{(2n+\mu+\nu-1)(2n+\mu+\nu+1)}}\delta_{n,m+1}\right.$$
$$\left.+\frac{1}{2n+\mu+\nu+2}\sqrt{\frac{(n+1)(n+\mu+1)(n+\nu+1)(n+\mu+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+3)}}\delta_{n,m-1}\right] \tag{B22}$$

The final constraint is suggested by the property (A4), that is, G(y) must be linear function of y to make (B20) fully symmetric and tridiagonal, i.e. $G(y)=\rho y+\sigma$. This makes (B20) takes the following form:

$$J_{n,m}=2\lambda\left\{\begin{array}{l}\left[\sigma+\dfrac{\rho\left(\nu^2-\mu^2\right)+2nc(\nu-\mu)(n+\mu+\nu+1)}{(2n+\mu+\nu)(2n+\mu+\nu+2)}\right]\delta_{n,m}\\ +\left[\begin{array}{l}\dfrac{2\rho-c(\mu+\nu+2)}{(2n+\mu+\nu+2)}\sqrt{\dfrac{(n+1)(n+\mu+\nu+1)(n+\mu+1)(n+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+3)}}\delta_{n,m-1}\\ +\dfrac{2\rho-c(\mu+\nu+2)}{(2n+\mu+\nu)}\sqrt{\dfrac{n(n+\mu+\nu)(n+\nu)(n+\mu)}{(2n+\mu+\nu-1)(2n+\mu+\nu+1)}}\delta_{n,m+1}\end{array}\right]\end{array}\right\}, \tag{B23}$$

B.4: The J-matrix in Laguerre Basis

Similarly to the Jacobi basis, we expand the spinor component in Laguerre basis:

$$\phi_n^+(y)=A_n y^\alpha e^{-\beta y}L_n^\nu(y), \tag{B25}$$

In order to write wave operator in this basis, we shall derive (B25) twice, which gives:

$$\frac{d\phi_n^+}{dy}=A_n y^\alpha e^{-\beta y}\left\{\frac{\alpha}{y}-\beta+\frac{d}{dy}\right\}L_n^\nu(y), \tag{B26}$$

$$\frac{d^2\phi_n^+}{dy^2}=A_n y^\alpha e^{-\beta y}\left\{\frac{\alpha(\alpha-1)}{y^2}-\frac{2\alpha\beta}{y}+\beta^2+2(\frac{\alpha}{y}-\beta)\frac{d}{dy}+\frac{d^2}{dy^2}\right\}L_n^\nu(y), \tag{B27}$$

Using these derivatives in the general form of the wave operator in (B12), we get:

$$\begin{aligned} J_{n,m} &= -\frac{A_n A_m}{\lambda} < L_n^\nu \,|\, y^{2\alpha} e^{-2\beta y} \left(1+\frac{Q_-}{2\lambda}\right) y'^2 \left[\frac{d^2}{dy^2} + \left(\frac{2\alpha+a}{y} - 2\beta + b\right)\frac{d}{dy}\right] |\, L_m^\nu > \\ &\quad - \frac{A_n A_m}{\lambda} < L_n^\nu \,|\, y^{2\alpha} e^{-2\beta y} y' \left[R - W + y'\frac{\dot{Q}_-}{2\lambda}\right]\frac{d}{dy} \,|\, L_m^\nu > \qquad , \\ &\quad + \frac{A_n A_m}{\lambda} < L_n^\nu \,|\, y^{2\alpha} e^{-2\beta y} F(y) \,|\, L_m^\nu > + n \leftrightarrow m \end{aligned} \tag{B28}$$

Where the coordinate transformation is taken such that $\frac{dy}{dx} = \lambda y^a e^{by}$, which gives $y'' = y'^2\left(\frac{a}{y} + b\right)$, for real parameters a and b. The function F(y) is given below:

$$\begin{aligned} F(y) &= \frac{\lambda Q_+}{2} - \left(R - W + y'\frac{\dot{Q}_-}{2\lambda}\right)\left[R + y'\left(\frac{\alpha}{y} - \beta\right)\right] + \left(1+\frac{Q_-}{2\lambda}\right)\times \\ &\quad \left\{R^2 - y'\dot{R} - y'^2\left[\frac{\alpha(\alpha+a-1)}{y'^2} - \frac{2\alpha\beta + \beta a - \alpha b}{y} + \beta(\beta - b)\right]\right\}, \end{aligned} \tag{B29}$$

In the next steps, we will use the properties of the Laguerre polynomials to make the wave operator tridiagonal and symmetric in this basis. First of all, if we compare the first line in (B28) with the properties (A6) and (A9), we get:

$$\begin{aligned} 2\alpha + a &= \nu + 1 \\ 2\beta - b &= 1 \end{aligned} \quad , \tag{B30}$$

The second line in (B28) is compared to (A9) to give:

$$R - W + y'\frac{\dot{Q}_-}{2\lambda} = cy' , \tag{B31}$$

Using the constraints in (B30) and (B31), we can write (B28) in a closed form:

$$J_{n,m} = c\lambda\left[-2n\delta_{n,m} + \sqrt{n(n+\nu)}\delta_{n,m+1} + \sqrt{(n+1)(n+\nu+1)}\delta_{n,m-1}\right] + 2\lambda A_n A_m \int_0^\infty y^\nu e^{-y} G(y) L_n^\nu L_m^\nu dy \,, \tag{B32}$$

where,

$$G(y) = \frac{\lambda Q_+}{2}\frac{y}{y'^2} - c\left(\alpha - \beta y + \frac{y}{y'}R\right) + \left(1 + \frac{Q_-}{2\lambda}\right) \times \left[\frac{y}{y'^2}\left(R^2 - y'\dot{R}\right) + \left(n + \frac{\nu + ab + 1}{2}\right) - \frac{1}{4}\frac{\nu^2 - (a-1)^2}{y} + \frac{b^2 - 1}{4}y\right], \tag{B33}$$

The property (A8) suggests that G(y) must be a linear function of y to achieve the requirement that the wave operator becomes tridiagonal and symmetric. To satisfy this, one should look for specific values of the potentials $\{V, S, W\}$ and the transformation parameters $\{a, b\}$. This is basically the procedure that we will follow to extract the solvable class of potentials that lie within this space where J is tridiagonal and symmetric. With $G(y) = \rho y + \sigma$, we can write (B32) in the following compact form:

$$J_{n,m} = 2\lambda\left\{\left[\sigma + 2\left(n + \frac{\nu+1}{2}\right)\rho - cn\right]\delta_{n,m} + \left(\frac{c}{2} - \rho\right)\left[\sqrt{n(n+\nu)}\delta_{n,m+1} + \sqrt{(n+1)(n+\nu+1)}\delta_{n,m-1}\right]\right\}, \tag{B.34}$$

Appendix C: Studying the Bound and Scattering States

Studying the potential energy is a little bit hard since we have different potentials in the problem, {V, S, W}. The only way to study these potentials is to investigate wether the system has bound states or scattering states or both is by transforming Dirac equation into Schrodinger equation for each spinor component. Making a general formalism is not that easy but we do this for some special cases. First of all, if we consider the case where S=V=0, and W is nonzero, then Dirac equation can be broken into two Schrodinger equations for each spinor component which are given below:

$$-\frac{1}{2m}\frac{d^2\overset{+}{\psi}(x)}{dx^2}+\frac{1}{2m}\left\{W^2-\frac{dW}{dx}\right\}\overset{+}{\psi}(x)=E\overset{+}{\psi}(x), \tag{C1}$$

$$-\frac{1}{2m}\frac{d^2\overset{-}{\psi}(x)}{dx^2}+\frac{1}{2m}\left\{W^2+\frac{dW}{dx}\right\}\overset{-}{\psi}(x)=E\overset{-}{\psi}(x), \tag{C2}$$

So, to study the energy states we should plot the potential $\left(W^2 \pm \frac{dW}{dx}\right)$ and determine wether it has bound states or scattering states or both.

Another case we consider here is the spin symmetric coupling when S=V, W is nonzero. Dirac equation will be broken into two Schrodinger equations as follows:

$$-\frac{1}{2m}\frac{d^2\overset{+}{\psi}(x)}{dx^2}+\frac{1}{2m}\left\{4mV+W^2-\frac{dW}{dx}\right\}\overset{+}{\psi}(x)=E\overset{+}{\psi}(x),$$

(C3)

$$-\frac{1}{2m}\frac{d^2}{dx^2}\chi^-(x)+\frac{1}{2m}\left[W'+\frac{2WV'}{(E-2V)}+W^2-\frac{2V''}{(E-2V)}\right]\chi^-(x)=\chi^-(x), \tag{C4}$$

where,

$\overset{-}{\psi}(x)=e^{f(x)}\chi^-(x)$ and $f(x)=-\int_0^x\frac{2V'(\xi)}{\left(E-2V(\xi)\right)}d\xi$.

So, we must study the following potentials to determine wether the relativistic problem has bound states or scattering states or both:

$$\tilde{V}_1(x,m) = 4mV + W^2 - \frac{dW}{dx}, \tag{C5}$$

$$\tilde{V}_2(x,E) = W' + \frac{2WV'}{(E-2V)} + W^2 - \frac{2V''}{(E-2V)}, \tag{C6}$$

Appendix D: Dirac Equation in Graphene

We are not interested here in talking about the physics of graphene to stay within our research topic but for those who are interested you can read different references [19, 20]. For the moment, we will focus on the massless Dirac equation in graphene and how to convert it to be identical to our Dirac equation. We start here by considering the Hamiltonian that describes the dynamics of Dirac fermions that move with Fermi velocity v_F:

$$H = v_F \vec{\sigma}.\vec{p} + U(x), \tag{D1}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y)$ are Pauli matrices and $U(x)$ is the potential function.

If we consider 1D symmetry in the spinor wavefunction, that is, $|\Psi(x,y)\rangle = e^{ik_y y}|\psi(x)\rangle$, then we can write Dirac equation associated with the above Hamiltonian as:

$$\begin{pmatrix} V(x) & -i\left(\frac{d}{dx}+k_y\right) \\ -i\left(\frac{d}{dx}-k_y\right) & V(x) \end{pmatrix}\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = \varepsilon \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}, \tag{D2}$$

where: $V(x) = U(x)/\hbar v_F$ and $\varepsilon = E/\hbar v_F$ (E is the energy eigenvalue of Dirac fermions).

Considering the transformation $\psi^- \to -i\psi^-$, we write (D2) as:

$$\begin{pmatrix} V(x) & -\frac{d}{dx}-k_y \\ \frac{d}{dx}-k_y & V(x) \end{pmatrix}\begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} = \varepsilon \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}, \tag{D3}$$

Equation (D3) can be transformed to our Dirac equation using unitary transformation $e^{-i\frac{\pi}{4}\sigma_y}$ which makes (D3) read as:

$$\begin{pmatrix} V(x)+k_y & -\frac{d}{dx} \\ \frac{d}{dx} & V(x)-k_y \end{pmatrix}\begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix} = \varepsilon \begin{pmatrix} \chi^+ \\ \chi^- \end{pmatrix}, \tag{D4}$$

where: $\chi^{\pm} = e^{-i\frac{\pi}{4}\sigma_2}\psi^{\pm}$.

Equation (D4) is identical to our Dirac equation with zero pseudopotential and the mass is replaced by the constant wavenumber k_y, more details can be found in [21]. The treatment of equation (D4) will fall within Jacobi basis since the potentials in this case are basically confined in the x-directions, i.e. the potential is defined over a finite interval of the real line.

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Vitae

Personal Information

Name	Ibsal A. T. Assi
Nationality	Palestinian
Date of Birth	30/03/1990
Email	asiebsal@gmail.com
Address	Ramallah, Palestine
Academic Background	Master Degree in Physics

EDUCATION

- King Fahd University of Petroleum and Minerals, Saudi Arabia. 2014-2016
  - Master of Science in Physics. GPA (3.81/4.00)
- Birzeit University, Palestine. 2008-2013
  - Bachelor of Science in Physics (with minor in Mathematics).

CONFERENNCES

- 5th Saudi International Meeting on Frontiers of Physics, 2016, Jazan University, KSA. (20 minute talk).

PUBLICATIONS

- Alhaidari, A. D., H. Bahlouli, and I. A. Assi. "Solving Dirac equation using the tridiagonal matrix representation approach." *Physics Letters A* (2016).

COMPUTER SKILLS

- I'm excellent in basic computer skills specially the excel and other Microsoft skills
- I'm very good in computational skills especially on Wolfram Mathematica

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LANGUAES

- Arabic (Mother tongue)
- English (Fluent)

REFERENCES

Dr. Hocine Bahlouli	Professor	bahlouli@kfupm.edu.sa
Dr. Watheq Al-Basheer	Assistant Professor	Watheq@kfupm.edu.sa
Dr. Fatah Khiari	Associate Professor	khiari@kfupm.edu.sa
Dr. Khali Ziq	Professor	kaziq@kfupm.edu.sa
Dr. Saeed Al-Marzoug	Associate Professor	marzoug@kfupm.edu.sa
Dr. Abdulaziz Alhaidari	Professor	haidari@sctp.org.sa